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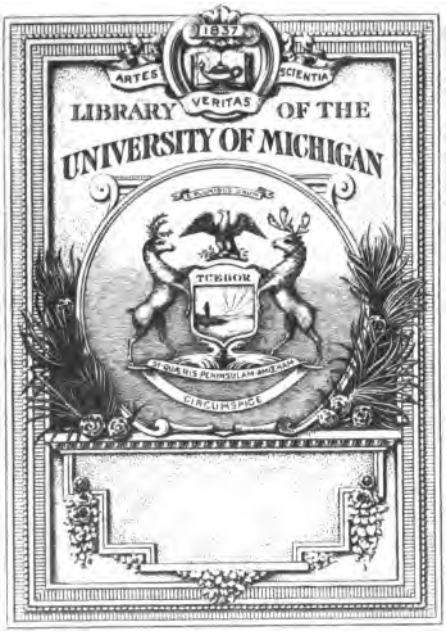

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By E. J. Townsend, Professor and Head of the Department of Mathematics in the University of Illinois.

HENRY HOLT AND COMPANY
NEW YORK CHICAGO

SCHOOL ALGEBRA

SECOND COURSE

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NEW YORK
HENRY HOLT AND COMPANY
1915

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PREFACE

This Algebra is the second volume of a series of two books. It contains material for a half-year course and presupposes a year's work such as is provided in the authors' *First Course*, or any similar text. Since, in many schools, a year or more intervenes between the first and second courses in algebra, it has been deemed advisable to include a review of the *First Course* before proceeding to new subjects. This part of the book is, however, more than a mere hasty resumé. While the student is reviewing the topics of the first year, he is at the same time making a distinct advance by seeing the subject from new view-points, which his added maturity and training enable him to appreciate.

The selection of new topics in this *Second Course* is in accord with current practice in the best high schools. While the book contains material for thorough preparation for college, the authors have not had in mind solely the students making such preparation, but rather the great body of students who will go no further in our educational system than the high school. The chapter devoted to the notion of a function is introduced in the belief that the high school student should become somewhat familiar with the idea of the correspondence between two related variables. Along with the notion of a function, the graph is presented with a view to leading the students to picture and to visualize this correspondence.

A special effort has been made to obtain interesting problems that have an inherent value even when considered apart from the algebraic principles involved in their solution. Without aiming to teach physics or engineering, some of the problems

have a physical setting; but, in such cases, care has been taken to assume no more technical knowledge than is possessed by the average high school student. A feature of the numerous review exercises and problems is the inclusion of some typical college entrance examination questions.

The authors wish to express their thanks to the teachers who have contributed to this book by their suggestions. We are especially indebted to Prof. E. J. Townsend and Dr. E. B. Lytle of the University of Illinois and to Miss Jessie D. Brakensiek of Quincy, Ill., High School, for careful and critical reading of the manuscript and for suggestions as to exercises and problems; to Mr. J. L. Dunn of Lewis and Clark High School, Spokane, Wash., Mr. C. H. Fullerton and Mr. W. B. Skimming of East High School, Columbus, O., Mr. E. A. Hook of the Commercial High School, Brooklyn, N. Y., Mr. L. C. Irwin of the Township High School, Joliet, Ill., and Mr. R. L. Modesitt of the Eastern Illinois State Normal School, Charleston, Ill., for reading the proof and seeing the book through the press.

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SCHOOL ALGEBRA

SECOND COURSE

A LIST OF SIGNS AND SYMBOLS

$+$, read *plus*. $-$, read *minus*.

$\times$, or $\cdot$, read *times*.

$\div$, read *divided by*.

$=$, read *is equal to*.

$\equiv$, read *is identical with*.

$\neq$, read *is not equal to*.

$<$, read *is less than*.

$>$, read *is greater than*.

$\leqq$, read *is less than or equal to*.

$\geqq$, read *is greater than or equal to*.

$a!$ or $\lfloor a$, read *factorial* **a**.

( ) Parentheses.
[ ] Brackets.
{ } Braces.
——— Vinculum.

Signs of aggregation. These signs are used to collect together symbols which are to be treated in operations as one symbol.

a_r, read **a** *subscript* **r**, or **a** *sub* **r**.

x', x'' ..., read **x** *prime*, **x** *second*, ... respectively.

$a : b$, read **a** *is to* **b**.

$\lim x$, read *limit of* **x**.

$x = \infty$, read **x** *becomes infinite*.

$\log_a n$, read *logarithm of* **n** *to the base* **a**.

a^n, read **a** *to the* **n***th power*, or **a** exponent **n**.

$\sqrt{a}$, read *square root of* **a**.

$\sqrt[n]{a}$, read **n***th root of* **a**.

$f(x)$, F (x), etc., read "**f**" *function of* **x**, "F" *function of* **x**.

(x, y), or $P(x, y)$ read *point whose coordinates are* **x** *and* **y**.

SCHOOL ALGEBRA

SECOND COURSE

CHAPTER I

ALGEBRAIC OPERATIONS

1. Four fundamental operations. In the first course in algebra, it has been seen that operations with numbers are made up of additions, subtractions, multiplications and divisions. These operations are called the four fundamental operations of algebra.

EXERCISES

1. If $a = 2$, name the fundamental operations used in finding the value of $a^2 + 2a + 1$; of $a - \frac{4}{a} + 2$.

2. What fundamental operations are used in finding the value of $\frac{x^3 - 3x^2 + 3x - 1}{x + 2}$ when $x = 2$?

3. The area of a trapezoid is $\frac{1}{2}(b + b')h$, where b and b' are the bases and h is the altitude. What operations are used in finding the area when $b = 12$, $b' = 10$, $h = 8$?

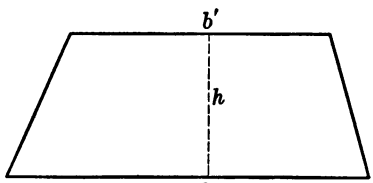

FIG. 1

4. The safe load W that may be put on a pile so that the pile will not sink under it is given by the formula,

$$W = \frac{2wh}{k + 1},$$

where W is the safe load in pounds, w is the weight of the pile driver hammer, h the distance in feet the hammer falls, and k the number of inches the pile goes in at each stroke of the hammer. What fundamental operations are involved in this formula?

2. Addition and subtraction. *The sum of two or more numbers is the same in whatever order they are added.* Thus,

$$a + b = b + a.$$

This is the **law * of order** or the **commutative law,** for addition.

The sum of two or more numbers is the same in whatever manner the numbers are grouped. Thus,

$$a + b + c = (a + b) + c = a + (b + c).$$

This is the **law of grouping** or the **associative law,** for addition.

Example 1. Add $5a + 2x$, $4x - 6a$, $8a - x$.

The sum may be written thus:

$$(5a + 2x) + (4x - 6a) + (8a - x). \qquad (1)$$

By the associative and commutative laws, we may write (1) in the form

$$5a + 8a - 6a + 2x + 4x - x. \qquad (2)$$

Again, by the associative law, (2) may be written as

$$7a + 5x.$$

This example illustrates the fact that polynomials may be added by combining similar terms. For this purpose, it is well to write the polynomials so that similar terms are in the same column.

This process applied to Example 1, gives:

$$\begin{array}{r} 5a + 2x \\ -6a + 4x \\ 8a - x \\ \hline 7a + 5x \end{array}$$

* These laws of algebra are not to be proved. They are in the nature of useful assumptions that have a wide application in experience.

Definition of subtraction. *To subtract a number* a *from a number* b *is to find a number which added to* a *will give the sum* b.

Example 2. Subtract $5x^2 - 6x + 4ax$ from $8x^2 - 4ax + 2x$.

PROCESS:

$$\begin{array}{r} 8x^2 - 4ax + 2x \\ 5x^2 + 4ax - 6x \\ \hline 3x^2 - 8ax + 8x \end{array}$$

3. Use of parentheses. In order to group terms together, we use parentheses. It should be remembered that parentheses may be removed with or without change of sign of each term included, according as the sign $-$ or $+$ precedes the parentheses.

Thus,
$$a - (b - c) = a - b + c,$$
$$a + (b - c) = a + b - c.$$

In case several parentheses or signs of aggregation are involved together, one within another, they may be removed by removing the innermost ones in turn, according to the rule just stated. Thus,

$$\begin{aligned} & a - \{3x - 4[x - (y - 5) - (2y + 6)]\} \\ = {} & a - \{3x - 4[x - y + 5 - 2y - 6]\} \\ = {} & a - \{3x - 4x + 12y + 4\} \\ = {} & a + x - 12y - 4. \end{aligned}$$

EXERCISES

Add:

1. $4ax$, $-2ax$, $6ax$.
2. $4x + 3y$, $2x - 6y$.
3. $5cx + 6a$, $4cx - 8a$.
4. $2x + 3y - 6z$, $3x - 8y + 7z$, $8x - 7y + 4z$.
5. $4a^2 + 5ac^2 - 3c^3$, $8a^2 - 7ac^2$, $2ac^2 + 5c^3$.
6. $2x^2 - 3xy$, $16x^2 + 17xy$.
7. $x^2 + 4ax + c$,
   $3x^2 - 2ax + 3c$,
   $x^2 - ax - 2c$.

Subtract the first expression from the second in the following:

8. $3a + b$, $4a - b$.

9. $2x^2 + 3x - 4$, $4x^2 - x + 5$.

10. $3x^2 + 4xy - 6y^2 - 6x$, $5x^2 - xy - y^2 + 6x$.

11. $4a^3 - 6ax^2 + 2ax - 3a + 4$, $2a^3 + ax - 8ax^2 + a - 6$.

12. $3(a - b)$, $5(a - b)$.

13. Show how you can check a subtraction by means of an addition.

14. From the sum of $6a - 7b + 4c$ and $8b - 4a + 2c$, subtract the sum of $2a - 5b + 2c$ and $6a - 3c + 5b$.

15. From the sum of $ax^2 + 3y + 4z$ and $3ax^2 - 8y + 10z$, subtract $2ax^2 + 7y + 10z$.

16. From the sum of $8(a - b) + 5(c - d)$ and $10(a - b) - 6(c - d)$, subtract $4(a - b) - 10(c - d)$.

Combine coefficients of similar terms:

17. $ax + bx + cx$.

SOLUTION: $ax + bx + cx = (a + b + c)x$.

18. $2ax - 6bx + 4cx$.

19. $4x - abx + cx$.

20. $m(a + b) + n(a + b)$.

SOLUTION: $m(a+b) + n(a + b) = (m+n)(a + b)$.

21. $m(a + b - c) + n(a + b - c) + l(a + b - c)$.

22. $3a(5m - 3n) + 5a(-3n + 5m)$.

Simplify by removing parentheses and combining like terms:

23. $4a - 5 + (y - 2a) + 3(y - a)$.

24. $8x + (3x - 5) - (6x - 11)$.

25. $6y - 2(y - 4) + 3(y - 5)$.

26. $2(x - 5) + 6(x + 1) - 3(x - 7)$.

27. $5x - 2[2 + 3(x - 1) + 5(x - 2) - 4(x - 5)]$.*

* For names of various symbols of aggregation, or of other symbols, see page facing Art. 1.

28. $5x - [2x - y - \{2x - y - (x - \overline{y - x}) + 2x\} - 3y]$.

29. $a - (-2b - c - \{a - b\}) - (5a - 2c - [-3b + c])$.

30. $z - (-z - \{-3z - [x - \overline{x + 3}] - 4\} - [2x - \overline{x - 3}])$.

31. $6x - 2[2 - 3(2x - 3 - a) - 5\{a - (2x - 2a) - 4\}]$.

32. $7a - [(3a - 4) - 6 - (3a - 4c - 2)]$.

33. $5x - 4[2(x - 4) + 3(2x - 1) - (x - 7)]$.

Simplify the following by combining like terms:

34. $6\sqrt{x^2 - 1} + 3\sqrt{x^2 - 1}$.

35. $5\sqrt{x^2 - a^2} + 2\sqrt{x^2 - a^2} - 3\sqrt{x^2 - a^2}$.

36. $m\sqrt{x^2 - a^2} + n\sqrt{x^2 - a^2}$.

37. $n(n - 1)(n - 2) + (n + 1)(n - 1)$

38. $n(n - 1)(n + 2) + n(n - 1)(n + 1) + n(n - 1)$.

In finding the value of this expression for $n = \frac{1}{2}$, is it simpler to evaluate the three terms separately or to combine them first?

39. State the rule for enclosing terms in parentheses preceded by a minus sign.

4. Multiplication. *The factors of a product may be taken in any order.* Thus,
$$ab = ba. \tag{1}$$

This is the **law of order** or the **commutative law,** for multiplication.

The factors of a product may be grouped in any manner. Thus,
$$abc = (ab)c = a(bc). \tag{2}$$

This is the **law of grouping** or the **associative law,** for multiplication.

To illustrate the use of the laws of order and grouping, we may write

$$\begin{aligned}(5ab)\ (3ab) &= (5 \cdot a \cdot b)(3 \cdot a \cdot b)\\ &= 5 \cdot 3(a \cdot a)(b \cdot b)\\ &= 15a^2b^2.\end{aligned}$$

5. Exponents. We write

$$a \cdot a \ldots . \text{ (to } m \text{ times)} = a^m,$$

and read a^m as "a exponent m" or the "mth power of a."

When m and n are positive integers,

$$\begin{aligned} a^m \cdot a^n &= a \cdot a \ldots (m \text{ times}) \cdot a \cdot a \ldots (n \text{ times}) \\ &= a \cdot a \cdot a \cdot \ldots (m + n \text{ times}) \\ &= a^{m+n}, \end{aligned}$$

or

$$a^m a^n = a^{m+n}. \tag{3}$$

That is, *the exponent of the product of two powers of a number is the sum of the exponents of the powers taken singly.* This is often called the first law of exponents.

ILLUSTRATIONS: $a^2 \cdot a^5 = a^7$, $3^2 \cdot 3^4 = 3^6$, $5 \cdot 5^2 \cdot 5^3 = 5^6$.

6. Multiplication of monomials. The main points involved in the multiplication of monomials are well illustrated by a simple example. Thus,

$$\begin{aligned} &(6a^2b^2)(-3a^2b)(4ab) \\ &= (6 \cdot -3 \cdot 4)(a^2 \cdot a^2 \cdot a)(b^2 \cdot b \cdot b) \quad [(1) \text{ and } (2), \text{ Art. 4.}] \\ &= -72a^5b^4. \end{aligned}$$

It is to be noted (1) that the coefficient in the product is the product of the coefficients in the given factors, (2) that the exponent of any letter in the product is the sum of the exponents of that letter in the given factors.

7. Multiplication of polynomials by monomials. It is a law of multiplication that

$$a(b + c + d + \ldots) = ab + ac + ad + \ldots \tag{1}$$

This is known as the **distributive law.**

According to this law, *to multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial and add the resulting products.*

Thus,
$$\begin{array}{r} 3a^2b^2 - 2ab^2 + 3a^2b \\ \hline 3ab \\ \hline 9a^3b^3 - 6a^2b^3 + 9a^3b^2. \end{array}$$

8. Multiplication of polynomials. *To find the product of two polynomials, we take the sum of the products obtained by multiplying each term of one polynomial by each term of the other.*

To illustrate, multiply $2x^2 + xy + 4y^2 + y$ by $x - y + xy$.

PROCESS:
$$\begin{array}{r} 2x^2 + xy \quad + 4y^2 \quad + y \\ \hline x \quad - y \quad + xy \\ \hline 2x^3 + x^2y + 4xy^2 + xy \\ - 2x^2y - xy^2 \quad - 4y^3 - y^2 \\ xy^2 \quad + 2x^2y + x^2y^2 + 4xy^3 \\ \hline 2x^3 - x^2y + 4xy^2 + xy - 4y^3 - y^2 + 2x^2y + x^2y^2 + 4xy^3. \end{array}$$

This process is, in the main, an application of the distributive law of multiplication.

9. Zero. The number zero may be defined by

$$0 = a - a,$$

where a is any number.

Then, it follows at once that *the product of any number k and 0 is 0.* For,

$$\begin{aligned} k \cdot 0 &= ka - ka \quad \text{(By the distributive law)} \\ &= 0. \qquad\quad \text{(By the definition of zero)} \end{aligned}$$

EXERCISES

Perform the multiplications indicated in the following:

1. $2^2 \cdot 3^4 \cdot 2^6 \cdot 3^5$.
2. $2ab^2 \cdot 3a^2b^2 \cdot 4a^3b$.
3. $4xy \cdot 3x^2y \cdot -2x^3y^2$.
4. $3abc \cdot -4ab^2c \cdot 2abc^2$.
5. $x^5 \cdot x^{10} \cdot x^{15}$.
6. $x^n \cdot x^{n+1} \cdot x^{n-1}$.
7. $2^{n+m+1} \cdot 3^{n-m-1} \cdot 2^{m-n-1} \cdot 3^{m+n-1}$.
8. $(1+x)^n \cdot (1-x)^m \cdot (1+x)^{m-n} \cdot (1-x)^n$.

9. $2a(4a^2 - 3a)$.
10. $5a^2(4ab + 3a^2b - 2a^3b^2)$.
11. $(2a^2 + 3ay)(3a^2 - 2ay)$.
12. $(a + b)(a + b)(a + b)$.
13. $(1 - x)(a + ax + ax^2 + ax^3)$.
14. $(a + b + c)^2$.
15. $(a + b - c)^2$.
16. $(a + b + c)(a + b - c)$.
17. $\left(2a - \frac{b}{2}\right)\left(2a + \frac{b}{2}\right)$.
18. $\left(2x - \frac{y}{3}\right)^3$.

10. Division. *To divide any number a by any number b ($b \neq 0$) * is to find a number x such that the product $bx = a$.*

Note the condition that $b \neq 0$. This means that the divisor is not zero.

From this definition of division,

$$\frac{a^2}{a} = a, \frac{a^4}{a^2} = a^2,$$

and in general

$$\frac{a^m}{a^n} = a^{m-n}, \tag{1}$$

where m and n are positive integers and $m > n$.

That is, *the exponent in the quotient of two powers of a number is the exponent of the dividend minus the exponent of the divisor.*

11. Division of monomials. The main points involved in the division of monomials may be brought out by a simple example.

Thus,

$$\frac{-6a^4b^2}{3a^2b} = -\frac{6}{3} \cdot \frac{a^4}{a^2} \cdot \frac{b^2}{b}$$
$$= -2a^2b.$$

It is to be noted (1) that the coefficient of the quotient is the quotient of the coefficients with due regard to the rule of signs,

* The sign $\neq$ is the sign of inequality and is read "is not equal to."

(2) that the exponent of any letter in the quotient is equal to the exponent of that letter in the dividend minus its exponent in the divisor.

12. Division of a polynomial by a monomial. According to the distributive law (Art. 6), a polynomial is divided by a monomial by dividing each term separately.

Thus, $$\frac{6x^2y - 3axy + 9axy^2}{3xy} = 2x - a + 3ay.$$

CHECK: Make $x = y = a = 1$. Dividend = 12.
Divisor = 3.
Quotient = 4.

But $$\frac{12}{3} = 4.$$

13. Division of a polynomial by a polynomial. The process of dividing one polynomial by another may be easily shown by examples. In general, the dividend and divisor should be arranged according to ascending or descending powers of some letter, called the letter of arrangement.

Example. Divide $a^4 - 7a^2 + 2a^3 - 8a + 12$ by $2 + a^2 - 3a$.

PROCESS:

$$\begin{array}{rl} a^4 + 2a^3 - 7a^2 - 8a + 12 & \mid a^2 - 3a + 2 \\ \underline{a^4 - 3a^3 + 2a^2} & \quad a^2 + 5a + 6 \\ 5a^3 - 9a^2 - 8a + 12 & \\ \underline{5a^3 - 15a^2 + 10a} & \\ 6a^2 - 18a + 12 & \\ \underline{6a^2 - 18a + 12} & \end{array}$$

Check this division by finding the product $(a^2 + 5a + 6)(a^2 - 3a + 2)$.

14. Zero in division. In the definition of division (Art. 10), it was stated that the divisor cannot be zero. That is to say, *division by zero is excluded from our operations in algebra.* When the dividend is zero, the quotient is zero; for,

$$\frac{0}{k} = 0,$$

since $$k \cdot 0 = 0. \quad \text{(Art. 9)}$$

Where is the fallacy in the following?

Let	$x = a.$	(1)
Multiply both sides by x,	$x^2 = ax.$	(2)
Subtract a^2 from both sides,	$x^2 - a^2 = ax - a^2.$	(3)
Divide both sides by $x - a$,	$x + a = a.$	(4)
But by (1)	$x = a.$	(5)
By (4) and (5)	$2a = a.$	(6)
Hence,	$2 = 1.$	(7)

EXERCISES

Divide:

1. $2^3 \cdot 3^4 \cdot 7^5$ by $2 \cdot 3^2 \cdot 7^3$.
2. $3a^7b^2c$ by a^2bc.
3. a^{m+2n} by a^n.
4. a^{c+3d+2} by a^{c+d+2}.
5. $a^{3n-5}x^{2n+1}$ by $a^{n-6}x^{n-1}$.
6. $(x-1)^{2n}(x+1)^{n+1}$ by $(x-1)^n(x+1)^n$.
7. $(x-y)^{10}(x+y)^7$ by $(x-y)^6(x+y)^5$.
8. $(a^2-b^2)^{2n+3}$ by $(a^2-b^2)^{2n}$.
9. $x^3y^3 - 5x^2y^4 - x^2y^2$ by $-x^2y^2$.
10. $vs^2 - 2gsv - 2s^2v$ by vs.
11. $\frac{1}{2}gt^2 - 3g$ by $\frac{1}{4}g$.
12. $12x^2t^2 + 9x^4t^3 - 15x^3t^2$ by $3x^2t^2$.
13. $9ax^3w - 15ax^2w^2 - 3ax^4w^3$ by $3ax^2w$.
14. $8ap^3q^5 + 24bp^2q^3 - 32p^3q^2$ by $8p^2q^2$.
15. $-15x^2yz - 5xy^2z - 15xyz^2$ by $5xyz$.
16. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a^2 + 2ab + b^2$.

Check your result by making $a = 1$, $b = 1$.

17. $a^3 - a^2b - ab^2 + b^3$ by $a^2 - 2ab + b^2$.

Check your result by making $a = 2$, $b = 1$. Why can this result not be checked by making $a = 1$, b = 1?

18. $x^5 - y^5$ by $x - y$.
19. $x^6 + y^6$ by $x^2 + y^2$.
20. $4y^2 + 4xy + x^2 - 12yz - 6xz + 9z^2$ by $2y + x - 3z$.

Check Exercise 20 by making $x = 1$, $y = 2$, $z = 1$.

21. $x^4 + 3x^3 + 7x^2 + 8x + 6$ by $x^2 + x + 3$.

22. $(a - b)^2 - 3(a - b + 2$ by $a - b - 1$.

23. $x^4 - y^4$ by $x - y$.

Check the result by making $x = 2$, $y = 1$. Why can the result not be checked by making $x = y = 1$?

24. $a^4 + 4ab^3 - 4abc + 3b^4 + 2b^2c - c^2$ by $a^2 - 2ab + 3b^2 - c$.

25. A man owns three farms whose values are

$$3x^4 + 5x^3 + 2x^2 + x + 5,$$
$$5x^4 + 2x^3 + 6x^2 + 8x - 7,$$

and

$$4x^4 + 3x^3 + 7x^2 + 2x + 2.$$

Find the value of the three together. He divides the land into parts of equal value among his x sons. What is the value of the land that each obtains?

26. Divide $\pi R^2 - \pi r^2$ by $\pi R + \pi r$.

15. Powers and roots. In exercises under Art. 4–11, we have performed operations with integral powers by means of the laws of exponents

$$a^m a^n = a^{m+n}, \tag{1}$$

and

$$\frac{a^m}{a^n} = a^{m-n}. \tag{2}$$

We shall soon find it convenient to use a third law of exponents

$$(a^m)^n = a^{mn} \tag{3}$$

To illustrate the law, $(3^4)^2 = 3^4 \cdot 3^4 = 3^8$.

To derive this law, we note that

$$\begin{aligned}(a^m)^n &= a^m \cdot a^m \cdot a^m \ldots \text{ to } n \text{ times} \\ &= a^{m+m+m+\cdots \text{ to } n \text{ terms}} \\ &= a^{mn}.\end{aligned}$$

The pupil knows something about square root and cube root. We may now well extend the notion of a root by defining the nth root of a number, where n is any positive integer.

DEFINITION. An **nth root** of a given number is a number whose nth power is equal to the given number.

For a given number x, the nth root is written $\sqrt[n]{x}$. When n is 2 the root is called a square root. When n is 3 the root is called a cube root. The number n is called the index of the root.

From (3), we note that the nth root of a^{mn} is a^m; for, a^m taken n times as a factor gives a^{mn}. Thus, $\sqrt{x^8} = \sqrt{x^4 \cdot x^4} = x^4$, $\sqrt[4]{x^{12a}} = \sqrt[4]{x^{3a \cdot 4}} = x^{3a}$, $\sqrt[5]{x^{10}} = x^2$, $\sqrt[4]{(x^2 - a^2)^4} = x^2 - a^2$.

In general

$$\sqrt[n]{a^{mn}} = a^m.$$

This may be stated in words as follows: *The root of a power is found by dividing the exponent of the power by the index of the root.*

16. Algebraic operations. An operation that involves only a definite number of the four fundamental operations and of the operations of finding powers and roots is often called an **algebraic operation.**

EXERCISES

Extract the roots indicated in the following:

1. $\sqrt{16a^4}$.
2. $\sqrt{4a^{20}}$.
3. $\sqrt{4a^{2n}}$.
4. $\sqrt[3]{8a^3}$.
5. $\sqrt[3]{-8a^3}$.
6. $\sqrt{16a^{4n}}$.
7. $\sqrt[p]{a^{3p}}$.
8. $\sqrt[3]{a^{3p}}$.
9. $\sqrt[n]{a^{5n}b^n}$.
10. $\sqrt[4]{x^{24}}$.
11. $\sqrt{a^6b^8}$.
12. $\sqrt[3]{x^6y^9}$.

13. Expand $(a - b)^2$ and check the result when $a = 2, b = 1$.

14. Expand $(a - b)^3$ and check the result by making $a = 3$, $b = 2$.

MISCELLANEOUS EXERCISES

Simplify the following by performing indicated operations:

1. $1 - [-1 + \{-2 - (-1 + 6)\} - 1]$.
2. $4a - [-6y - (5 - 2y - 3x) + 4a] - [5 - (4y - 3x)]$.

3. $\dfrac{x^4 + x^2y^2 + y^4}{x^2 + y^2 - xy}$. Check result by making $x = y = 1$.

4. $\dfrac{\sqrt{x} + \sqrt{a}}{\sqrt{x} - \sqrt{a}} \cdot \dfrac{1}{\sqrt{x} + \sqrt{a}}$. Check result for $x = 9$, $a = 4$.

5. $(a + b)[a^2 - (ab - b^2)]$.

6. $\dfrac{6m^4 + m^3 - 29m^2 + 27m - 9}{3m^3 + 5m^2 - 7m + 3}$. Check result by making $m = 1$.

7. $\dfrac{1.2x^4 - 2.9x^3 + 0.9x^2 + x}{0.3x - 0.5}$.

8. $\dfrac{6x^{2m}}{3x^m}$.

9. $\dfrac{8x^{2m+n}}{4x^{m-n}}$.

10. $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$.

11. If $x = 3$, $y = 1$, $z = 0$, and $a = 1$, find the value of $x - (x - y) + \{x - (y + z)\} - [x - (y - \overline{z - a})]$.

12. If $x = m + n - 2p$, $y = m - 2n + p$, and $z = n + p - 2m$, show that $x + y + z = 0$.

13. If $a = x^3 + 4x^2 - 1$, $b = 1 - x - 2x^2$, and $c = 2x^3 + 2x^2 + x + 1$, find the value of $-a - [b - (2a - c) + c]$ when $x = 2$.

14. Find the value of $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ when $a = 4$, $b = 10$, and $c = 4$.

15. Find the value of $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ when $a = 4$, $b = 10$, and $c = 4$.

16. Divide $x^6 - y^6$ by $x^3 - y^3$, and check the result by making $x = 2$ and $y = 1$. Why can not the check be made for $x = 1$, $y = 1$?

17. Divide $x^{2m} - y^{2m}$ by $x^m + y^m$.

18. Divide $\pi R^2 - \pi r^2$ by $\pi(R - r)$.

19. Divide $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$ by $R - r$.

20. Divide $\frac{1}{6}\pi D^3 + \frac{1}{6}\pi d^3$ by $\pi(D + d)$.

Historical note on the fundamental laws. When the attention of the student is first called to these laws, it is not unlikely that he will regard

them as self evident propositions. That they are not self evident should be clear from the fact that systems of numbers have been devised, that do not follow the same laws of combination as apply to the numbers we use in algebra. In one prominent system, we have the product $ij = -ji$. While the study of kinds of numbers that do not obey the commutative, associative and distributive laws, belongs to advanced mathematics, the fact of the existence of such numbers is mentioned here to emphasize the fact that the laws of algebra are assumptions, even if they have a wide application in experience. Strange as it may seem at first thought, we find, upon examining the history of algebra, that much was known of certain complex processes before it was understood that laws about order and grouping are at the basis of these operations. Sir William Rowan Hamilton (1805–1865) stands out prominently as a man who helped to make clear the nature of algebraic processes by devising numbers that do not obey all of the laws given in this book.

CHAPTER II

LINEAR EQUATIONS IN ONE UNKNOWN

17. Equalities. A statement that one expression is equal to another expression is called an **equality.** The two expressions are called the **members** of the equality.

There are two kinds of equalities, **identical equalities** or **identities,** and **conditional equalities** or **equations.**

The two members of an identity are equal for all values of the symbols for which the expressions are defined.* Thus,

$$x^2 - a^2 = (x - a)(x + a),\ 5a = 10a - 5a,$$

are identities.

But in the equality

$$x - 5 = 4,$$

the two expressions $x - 5$ and 4 are equal only when $x = 9$. An equality of this kind, in which the members are equal only for particular values of the letters involved, that is, are not equal for all values, is sometimes called a conditional equality. In this book, we use the term equation to mean conditional equality. When it seems necessary to emphasize that an equality is an identity and not a conditional equality, we use the sign $\equiv$ instead of the sign $=$ between the members. But the sign $=$ will be used for both identities and equations when this usage can lead to no confusion.

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* This statement implies that we may not assign values to the letter which will make the members meaningless. Thus, $\frac{1}{1-x} = 1 + \frac{x}{1-x}$ is excluded when $x = 1$, since division by zero is excluded. (See Art. 14.)

EXERCISES

Which of the following equalities are identities?

1. $(x - a)^2 = x^2 - 2ax + a^2$.
2. $\dfrac{x^2 - a^2}{x - a} = x + a$.
3. $x^2 - 3x + 2 = 0$.
4. $x^2 + 6 = 0$.
5. $\dfrac{2}{1 - x} = 2 + 2x + \dfrac{2x^2}{1 - x}$.
6. $4x^2 + 4x + 1 = 0$.

18. Equations and identities as sentences. An equation may be regarded as essentially an interrogative sentence, asking for values of the letters that make the members equal. Thus,

$$x - 5 = 6$$

asks for the value of x that makes $x - 5 = 6$.

An identity is essentially a declarative sentence, stating the fact that two members are equal without regard to the values of the letters. Thus,

$$x^2 + 2ax = 2x^2 - x^2 + 3ax - ax$$

for any values that may be assigned to the letters.

19. Solution or root of an equation. To **solve** an equation in one unknown is to find values of the unknown that reduce the equation to an identity. Any such value is said to satisfy the equation and is called a **solution** or **root** of the equation.

20. Equivalent equations. Two equations are said to be equivalent when they have the same roots; that is, when each equation is satisfied by the solutions of the other. Thus, the equations

$$x - 2 = 0 \text{ and } 3x - 6 = 0$$

are equivalent.

21. Operations on equations. It is here taken for granted that in the first course in algebra the student has made considerable use of the following operations on members of an equation in the process of finding solutions:

(1) Adding the same number to both members

(2) Subtracting the same number from both members.

(3) Multiplying both members by the same number, other than zero.

(4) Dividing both members by the same number, other than zero.

These operations are permissible because they lead to equivalent equations. The operations (1) and (2) are often replaced by an equivalent operation called **transposition.** It consists in changing a term from one member of an equation to the other and changing the sign of the term.

The necessity for the restriction in (3) that the multiplier is not zero may be seen from the following examples:

Example 1. Given	$x - 1 = 3.$	(1)
Multiply by $x - 2$,	$x^2 - 3x + 2 = 3(x - 2).$	(2)
From (2)	$x^2 - 6x + 8 = 0.$	(3)
Whence,	$(x - 2)(x - 4) = 0.$	(4)

But, $x = 2$, and $x = 4$ satisfy (4) while (1) has the root 4 only.

The non-equivalence of (1) and (4) is due to the fact that the multiplier $x - 2$ is zero when $x = 2$. Such values as $x = 2$ which satisfy a certain derived equation, but do not satisfy the original equation, are sometimes called **extraneous** roots.

Example 2. Given	$x^2 - 3x + 2 = 3x - 6.$	(1)
Divide by $x - 2$,	$x - 1 = 3,$	(2)
and	$x = 4.$	(3)

But (1) has solutions $x = 2$ and $x = 4$, while (2) has only the solution $x = 4$.

The non-equivalence of (1) and (2) is due to the fact that we have made the error of dividing formally by $x - 2$, which is zero when $x = 2$. But division by zero is excluded from algebraic operations. (See Art. 14.)

22. Linear equations in one unknown. An equation of the form

$$ax + b = 0, \; a \neq 0 \tag{1}$$

is called a simple or linear equation in the unknown x. We have solved many such equations in the first course.

The general solution of (1) is

$$x = -\frac{b}{a},$$

as may be verified by substitution, but it is in general better to treat each case separately than to remember the formula.

23. Verification by substitution. The operations of Art. 21 are useful in finding solutions, but the solution is incomplete until the values of the unknown are substituted in the equation to be solved, and are shown to satisfy it.

EXERCISES AND PROBLEMS

Solve the following equations for x:

1. $3x + 6 = 5x + 10.$

SOLUTION:	$3x + 6 = 5x + 10.$	(1)
Transpose and collect,	$-2x = 4.$	(2)
Divide by -2,	$x = -2.$	(3)

CHECK. Substitute in (1), and

$$3(-2) + 6 = 5(-2) + 10.$$
$$0 = 0.$$

2. $\frac{x}{2} + \frac{x}{3} = \frac{5}{4} - x.$

SOLUTION:	$\frac{x}{2} + \frac{x}{3} = \frac{5}{4} - x.$	(1)
Multiply by 12,	$6x + 4x = 15 - 12x.$	(2)
Transpose and collect,	$22x = 15,$	(3)
	$x = \frac{15}{22}.$	(4)

CHECK. From (1) and (4)

$$\frac{15}{44} + \frac{5}{22} = \frac{5}{4} - \frac{15}{22},$$

or

$$\frac{25}{44} = \frac{25}{44}.$$

3. $7x + 9 = 2x - 6.$
4. $5x + 7 = 2x + 10.$
5. $5x + 7 = 2x + 9.$
6. $3x(2x + 1) = 6x^2 - 9.$
7. $3(x + 1) + x^2 = x^2 + 12.$
8. $5(x - 2) = 3(x + 1) - 13.$
9. $(x + 1)(x + 3) = x(x - 2).$
10. $ax + a = 6a.$
11. $3(a - x) = 9a.$
12. $2(x^2 - 2x + 2) - (x + 1) = 2x^2 + 10.$
13. $(x + 2)(x - 5) = (x + 4)(x - 1).$
14. $2[x + x(x - 1) + 1] = (2x - 1)(x + 2).$
15. $(m + n)x + (m - n)x = mn.$
16. $\frac{x}{2} + \frac{2x}{5} = \frac{3x}{10} + 12.$
17. $\frac{x}{a} + \frac{x + b}{b} = 1.$
18. $1.5x + 3.5x = 2.5x + 17.5.$
19. $0.1(5x + 20) - 0.4(x - 5) = 25.$
20. $\frac{5x - 8}{11} + \frac{2(10 - x)}{2} = x.$
21. $\frac{x}{a} + \frac{y}{b} = 1.$
22. $p(p - x) = y(y + x).$
23. $mx = m_1w_1 + m_2w_2 - nx.$*
24. $y - 5 = m(x - 10).$

* The subscript notation is found very convenient both in pure and in applied mathematics. The symbol m_1 is read "m sub 1."

The following exercises, 25–32, give relations in the notation of physics.

25. Given $s = vt$, solve for v.

26. Given $s = vt + s_0$, solve for v.

27. Given $s = \frac{1}{2}gt^2$, solve for g.

28. Given $s = \frac{1}{2}gt^2 + s_0$, solve for g.

29. Given $E = \frac{1}{2}Mv^2$, solve for M.

30. Given $D = \dfrac{W}{w_1 - w_2}$, solve for W.

31. Given $F = \frac{9}{5}C + 32$, solve for C.

32. Given $PV = pv\left(1 + \dfrac{t}{273}\right)$, express t in terms of P, V, p, v.

33. What extraneous root is introduced into an equation $x = 5$ if both members are multiplied by x? by $x - 2$? by $x + 3$?

34. What extraneous root is introduced into the equation $ax + b = 0$ if both members are multiplied (1) by x? (2) by $x - 2$? (3) by $x - a$?

35. What root is lost if the members of $x^2 - 4 = x - 2$ are divided by $x - 2$?

36. If each member of an equation is multiplied or divided by the same expression that does not contain the unknown, are the roots changed?

37. Five times a certain number when diminished by 23, is equal to 5 more than the number. What is the number?

38. A father is four times as old as his son and the sum of their ages is 65. What is the age of each?

39. Find three consecutive integers whose sum is 78.

40. What number must be added to each of the numbers 18, 108, and 48 in order that the product of the first two sums shall be equal to the square of the last sum?

41. What number must be added to each of the numbers a, b, and c, in order that the product of the first two sums shall be equal to the square of the last?

42. How soon after noon will the hands of a clock be together again?

SUGGESTION: Let x = number of minute spaces which the minute hand has traveled from noon until it overtakes the hour hand. Then $\frac{x}{12}$ will be the number of spaces the hour hand covers meanwhile. The difference, $x - \frac{x}{12}$ is 60 spaces. Why?

43. Denochares has lived a fourth of his life as a boy; a fifth as a youth; a third as a man; and has spent 13 years in his dotage. How old is he? (From the Collection of Problems by Methodorus, 310 A. D.)

44. A man can do a piece of work in 4 days, another in 6 days and a third in 12 days. How many days will it require all to do it when working together?

45. A room is 4 feet longer than it is wide, and if the length were decreased by 2 feet and its width increased by 4 feet, the area of the floor would be increased 40 square feet. Find the dimensions of the room.

PROBLEMS PERTAINING TO MIXTURES

46. A grocer mixes two grades of coffee which cost him 20 cents and 35 cents per pound respectively. How much of each must he take to make a mixture of 60 pounds which he can sell at 30 cents per pound with a profit of 20%?

47. A dealer has 2000 gallons of alcohol 85% pure. He wishes to add water until it is 75% pure. How much water must he add?

48. How much cream that contains 25% butter fat should be added to 1000 pounds of milk that contains $3\frac{3}{4}$% butter fat to produce a standard milk with 4% of butter fat?

49. How many pounds each of water and milk with 4% of butter fat should be taken to give a mixture of 160 pounds with $3\frac{1}{2}$% butter fat?

50. If 24 pounds of iron weigh 22 pounds in water and 20 pounds of lead weigh 19 pounds in water, find the amounts

(weights in air) of iron and lead in a mass which weighs 180 pounds in air and 170 in water.

51. A dealer mixes a pounds of tea worth x cents a pound with b pounds of tea worth y cents a pound and with c pounds of tea worth z cents a pound. Find the value, v, of the mixture in cents per pound.

PROBLEMS PERTAINING TO BUSINESS

52. The gross income of a certain man was \$110 more in the second of two years than in the first, but in consequence of an income tax of 1 % on the part of the income above \$3000, the net income (after paying tax) was the same in the two years. Find his income in each year.

53. A married man has a \$4000 exemption from income tax, but pays at the rate of 1 % on the rest of his income. He finds that after deducting income tax, his actual income is \$5980. On what amount does he pay the 1 % tax?

54. A man made two investments amounting together to \$6000. On the first he gained 10 %, and on the second he lost 5 %. His net gain on the two was \$60. What was the amount of each investment?

55. At a ball game the charge was 75 cents for each reserved seat and 50 cents for each general admission ticket. The ticket seller found he had sold 4320 tickets for \$2785. How many people bought reserved seat tickets?

PROBLEMS INVOLVING THE LEVER

Two boys, A and B, are balanced on a teeter board as shown in Fig. 2. The board is twelve feet long with the point of support at the middle. The boys find that they must sit at distances from the support such that the products obtained by multiplying the weight of each by his distance from the fulcrum or point of support are equal.

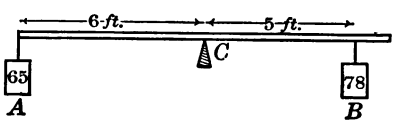

FIG. 2

Thus, $(78)(5) = (65)(6)$.

In the general problem of the lever the force F required to lift the weight W depends upon the position of the point of support (**fulcrum**) and the lengths of the arms of the lever. If f and w represent the distances of the force and weight respectively from the fulcrum, then the law of the lever is expressed by,

$$W \cdot w = F \cdot f.$$

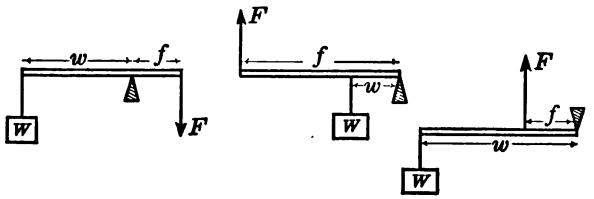

FIG. 3

56. A and B weigh 70 and 120 pounds respectively. On a teeter board, A is to sit 6 feet from the fulcrum and B 5 feet. A stone is placed on A's side 5 feet from the fulcrum so that the sides will balance. How heavy is the stone?

57. A and B together weigh 320 pounds. They balance when A is 8 feet from the fulcrum and B is 6 feet. Find the weight of each.

58. How heavy a stone can a man, by exerting a force of 200 pounds, lift with a crowbar 6 feet in length if the fulcrum be six inches from the stone (neglect weight of crowbar)?

PROBLEMS INVOLVING UNIFORM MOTION

The *rate* of uniform motion is the distance traversed in each unit of time. The rate of motion is often called the *speed* or *velocity*. It is customary to use s for distance or space traversed, v for velocity, and t for the time.

59. A Zeppelin flies 55 miles per hour and an aeroplane 70 miles per hour. If the aeroplane is 7.5 miles behind the Zeppelin, how long will it require to overtake the Zeppelin?

60. A freight train leaves New York for Chicago making a speed of 30 miles per hour. Five hours later a limited express leaves New York making 54 miles per hour. In what time will the express overtake the freight?

61. In how much time does the minute hand of a clock gain a complete revolution on the hour hand?

62. If the speed at which sound travels in air is given by

$$v = 1090 + 1.14(t - 32),$$

where v is the speed in feet per second, and t the temperature of the air expressed in Fahrenheit degrees, find the temperature of the air when sound travels 1120 feet per second.

CHAPTER III

FACTORING

24. Rational integral expressions. By a **rational integral expression** in certain letters, say in x, y, z, we mean the sum of terms of the type

$$ax^m y^n z^p,$$

where the exponents m, n, and p, are any positive integers.

Thus, $5x^3 + 10a^2 + 4x - 6,$

$$3xy + y^2\sqrt{5} - \frac{2}{3}x^2 - x\sqrt[3]{7},$$

and $ax^2y + 10xy^2 - cxyz$

are all rational integral expressions in all the letters involved, while $3x^3 + x\sqrt{y}$, and $x^2 + \frac{3x}{2y}$ are rational integral expressions in x but not in y.

It should be noted that a rational integral expression with respect to a letter x contains no indicated root of that letter, as $\sqrt{x}$, and does not have that letter in the denominator of a fraction.

The **degree** of a term of a rational integral expression is the sum of the exponents of the letters in the term. The degree of a rational integral expression is defined as that of a term whose degree is equal to or greater than that of any other term in the expression.

Thus, the degree of $3x^2y + 2x + y^2$ is 3.

We are especially concerned in this chapter with the factors of rational integral expressions.

25. Definition of factoring. We shall use the term **factor** to mean a rational integral factor.

A rational integral expression is **prime** when it has no factor besides itself and unity.

Finding the prime factors of a rational integral expression is called **factoring** the expression.

26. Important special products. Factoring in elementary algebra depends mainly upon the recognition of certain type products. The following have already been treated in the First Course. They should be reviewed and memorized.

a. Common monomial factor. $ax + ay = a(x + y)$.

Example. $2am - 4a^2 = 2a(m - 2a)$.

b. Difference of two squares. $a^2 - b^2 = (a + b)(a - b)$.

Example. $81x^2 - 1 = (9x + 1)(9x - 1)$.

c. Trinomial square. $a^2 + 2ab + b^2 = (a + b)^2$.

Example. $4x^2 - 12xy + 9y^2 = (2x - 3y)^2$.

d. Trinomial of the form $x^2 + (a + b)x + ab = (x + a)(x + b)$.

Example. $x^2 + 7x + 10 = (x + 5)(x + 2)$.

e. Sum of two cubes. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Example. $125x^6 + 8y^{15} = (5x^2 + 2y^5)(25x^4 - 10x^2y^5 + 4y^{10})$.

f. Difference of two cubes. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

Example. $m^3y^3 - 27n^9 = (my - 3n^3)(m^2y^2 + 3mn^3y + 9n^6)$.

g. Factors found by grouping.

$$ax + ay + bx + by = (a + b)(x + y).$$

Example.

$$2ax + 4bx - 3ay - 6by = 2x(a + 2b) - 3y(a + 2b) = (2x - 3y)(a + 2b).$$

h. Trinomial of form $ax^2 + bx + c$. Certain expressions of this form can be factored by inspection.

Example. Factor $6x^2 + x - 15$.

SOLUTION. The factors are two binomials whose first terms are $3x$ and $2x$ or $6x$ and x, and whose last two terms are ± 3 and ∓ 5, or ± 1 and ∓ 15, if the coefficients are integers. We must now choose the terms of the binomials so that the algebraic sum of the cross products is $+x$. By trial we find that the factors are $2x - 3$ and $3x + 5$.

EXERCISES

Factor the following:

1. $a + ab$.
2. $3 + 3x$.
3. $m^2 - 4$.
4. $a^2 + 10a + 25$.
5. $m^2 - 9m + 14$.
6. $a^2b^4 - x^4$.
7. $mp + mq + np + nq$.
8. $10am + 8aq + 15mx + 12qx$.
9. $5r^3 + 5$.
10. $k^2 + 6k - 55$.
11. $c^4 - 12c^2 + 36$.
12. $5p^2 - 9p - 18$.
13. $6ax^3 + 24ax^2 + 24ax$.
14. $2ax + 6a^2 - 4ay$.
15. $3t^3 + 81$.
16. $r^2 + 22r + 117$.
17. $x^2 + \frac{2}{3}x + \frac{1}{9}$.
18. $m^2n^4 - \frac{1}{4}$.
19. $3l^2 + 16l - 35$.
20. $am + bm + a + b$.
21. $a^{2r} + 20a^r + 100$. Find two factors only.
22. $x^{m+2} - 11x^{m+1} - 102x^m$.
23. $z^{3n} - 5z^{2n} + 4z^n$. Find three factors only.
24. $a^{2b} - 1$. Find two factors only.
25. $3x^2 - 3$.
26. $ab + a + b + 1$.
27. $3ax - 5by - 5ay + 3bx$.
28. $x^3 - 1$.

29. $m^4 - n^4$.

30. $(3x - y)(2a + 4) + 5(3x - y)$.

31. $(2a - 3b)^2 - 4b^2$.

32. $a^2 - b^2 + 2bc - c^2$.

33. $x^3 + 4x^2 + 4x$.

34. $a^2 - 7ab + 12b^2$.

35. $8x^3y^6 - 125m^9$. Find two factors only.

36. $3bmx + 2bm - 3pqx - 2pq$.

37. $1 - x^6$.

38. $60a^2 + 8ax - 3x^2$.

39. $a^4b + 5a^3b^2 + 4a^2b^3$.

40. $24m^4n^6y - 81my^4$.

41. $(m + 2)^2 - 5(m + 2) - 176$.

42. $5a^2 - 8ab - 4b^2$.

43. $7x^5 - 7x^3 - 84x$.

44. $x^2 - 4ax - 4b^2 + 8ab$.

45. $(a + b)(c^2 - d^2) - (a^2 - b^2)(c - d)$.

46. $2x^2 - 5x - 3$.

47. $21x^2 - 23x + 6$.

48. $(2x + a - b)^2 - (x - a + b)^2$.

49. $x^8 - x^6 - 64x^2 + 64$. Find six factors.

50. $x^6 + 16x^3 + 64$.

51. $12x^2 + 25x - 7$.

52. $15x^4 - x^3 - 6x^2$.

53. $9a^2x^4 - 30a^2x^3 + 25a^2x^2$.

54. $(x - 1)^3 - 9(x - 1)^2 + 8(x - 1)$.

55. $1 - x^2 - 2xy - y^2$.

56. $a^2c^2 + acd + abc + bd$.

57. $x^{2n} - 6x^n + 8$. Find two factors only.

58. $a^{4p} - 5a^{2p}b^q - 14b^{2q}$. Find two factors only.

59. $a^nb^{2n+2} - 16a^{3n}b^2c^{6m}$. Find four factors.

27. Further type products. The following type products are also useful and should be mastered.

i. Square of a polynomial. The square of a polynomial equals the sum of the squares of the terms plus twice the product of each term by every term that follows. Thus,

$$(a + b + c + d)^2 =$$
$$a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

If a polynomial can be put into the form of this product it can be factored.

Example.

$$\begin{aligned} & x^2 + 4y^2 + 9z^2 + 4xy + 6xz + 12yz \\ &= (x)^2 + (2y)^2 + (3z)^2 + 2 \cdot x \cdot 2y + 2 \cdot x \cdot 3z + 2 \cdot 2y \cdot 3z \\ &= (x + 2y + 3z)^2. \end{aligned}$$

j. Cube of a binomial. By performing the multiplication, we find that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

If a polynomial can be put into this form it can be factored.

Example. $8x^3 + 36x^2y + 54xy^2 + 27y^3$

$$\begin{aligned} &= (2x)^3 + 3 \cdot (2x)^2 \cdot (3y) + 3 \cdot (2x) \cdot (3y)^2 + (3y)^3 \\ &= (2x + 3y)^3. \end{aligned}$$

The student should find the cube of $a - b$.

k. Expressions that can be reduced to the form of the difference of two squares.

Example 1. Factor $x^4 + x^2y^2 + y^4$.

By the addition and subtraction of x^2y^2 we have,

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy). \end{aligned}$$

Example 2. Factor $36a^4 + 44a^2b^2 + 25b^4$. Find two factors only.

If $16a^2b^2$ be added to $36a^4 + 44a^2b^2 + 25b^4$, the resulting expression is a perfect square. Adding and subtracting $16a^2b^2$, we have

$$\begin{aligned} 36a^4 + 44a^2b^2 + 25b^4 &= 36a^4 + 60a^2b^2 + 25b^4 - 16a^2b^2 \\ &= (6a^2 + 5b^2)^2 - (4ab)^2 \\ &= (6a^2 + 5b^2 - 4ab)(6a^2 + 5b^2 + 4ab). \end{aligned}$$

EXERCISES

Factor the following:

1. $x^2 + y^2 + 16z^2 + 2xy + 8xz + 8yz$.
2. $4a^2 + 9b^2 + c^2 - 12ab + 4ac - 6bc$.
3. $x^3 + 6x^2y + 12xy^2 + 8y^3$.
4. $8m^3 - 36m^2n + 54mn^2 - 27n^3$.
5. $x^4 + x^2 + 1$.
6. $x^4 - 3x^2 + 1$.
7. $a^4 + 4a^2 + 16$.
8. $a^3 - 6a^2b + 12ab^2 - 8b^3$.
9. $x^9y^6z^3 + 6x^6y^4z^2 + 12x^3y^2z + 8$.
10. $x^4 - 11x^2 + 1$.
11. $x^2 + 9y^2 - 6xy + 6y - 2x + 1$.
12. $3 + 3x^2 + 3x^4$.
13. $x^6 - 3x^4 + 3x^2 - 1$.
14. $x^6 + y^4 + 2x^3y^2 + 2y^2z + z^2 + 2x^3z$.
15. $n^2 + p^2 + q^2 + 2np + 2nq + 2pq + 4(n + p + q)$.
16. $x^4 + 2x^3 + 3x^2 + 2x + 1$.
17. $a^3x^6 - 3a^3x^4 + 3a^3x^2 - a^3$.
18. $4a^4 + 12a^3b + 12a^2b^2 + 4ab^3$.
19. $8a^{3m} - 12a^{2m}b^n + 6a^mb^{2n} - b^{3n}$.
20. $x^{4m} + x^{2m} + 1$. Find two factors only.

28. Factor theorem. Divide $3x^2 - 10x + 7$ by $x - a$.

$$\begin{array}{l|l} 3x^2 - 10x + 7 & x - a \\ \underline{3x^2 - 3ax} & \overline{3x + 3a - 10} \\ (3a - 10)x + 7 & \\ \underline{(3a - 10)x - 3a^2 + 10a} & \\ 3a^2 - 10a + 7 & \end{array}$$

The remainder is seen to be the same as the dividend when a is substituted for x. This exercise illustrates the principle that *the remainder resulting from dividing a rational integral expression in x by $x - a$ may be obtained by substituting a for x in the given expression.*

To find the remainder when the above expression is divided by $x - 1$, we substitute 1 for x. This gives

$$3 \cdot 1^2 - 10 \cdot 1 + 7 = 0.$$

Since the remainder is zero, the expression is divisible by $x - 1$; that is, $x - 1$ is a factor of $3x^2 - 10x + 7$.

This exercise illustrates the important principle known as the

FACTOR THEOREM.* *If a rational integral expression in x becomes zero when a is substituted for x, then $x - a$ is a factor of the expression.*

The factor theorem gives a simple means of factoring certain expressions.

Example 1. Factor $5x^2 - 13x + 6$.

We wish to find a factor of the form $x - a$. In order that the given expression be exactly divisible by $x - a$, where a is an integer, a must be a factor of 6, that is, a must be ± 1, ± 2, ± 3, or ± 6. If we substitute 1 for x we have,

$$5 \cdot 1^2 - 13 \cdot 1 + 6 = -2.$$

If we substitute -1 for x we have,

$$5 \cdot (-1)^2 - 13 \cdot (-1) + 6 = 24.$$

If we substitute 2 for x we have,

$$5 \cdot 2^2 - 13 \cdot 2 + 6 = 0.$$

Hence $x - 2$ is a factor of the given expression. The other factor is found by division to be $5x - 3$, and we have

$$5x^2 - 13x + 6 = (x - 2)(5x - 3).$$

Example 2. Factor $3x^2 + 4x - 15$.

By substituting -3 for x we have,

$$3 \cdot (-3)^2 + 4 \cdot (-3) - 15 = 27 - 12 - 15 = 0.$$

Hence $x - (-3) = x + 3$ is one factor. The other factor is found by division to be $3x - 5$, and we have

$$3x^2 + 4x - 15 = (x + 3)(3x - 5).$$

* For the expression $ax^2 + bx + c$, a proof is given on p. 130. For proof of the general theorem see Rietz and Crathorne, College Algebra, p. 121.

Example 3. Factor $x^3 - 2x^2 - 5x + 6$.

By the use of the factor theorem one factor of the given expression is found to be $x - 1$. The quotient found by dividing $x^3 - 2x^2 - 5x + 6$ by $x - 1$ is $x^2 - x - 6$. The factors of this quotient are $x + 2$ and $x - 3$. Hence

$$x^3 - 2x^2 - 5x + 6 = (x - 1)(x + 2)(x - 3).$$

EXERCISES

Factor the following expressions:

1. $x^2 - 3x + 2$.
2. $3x^2 - 9x + 6$.
3. $2x^3 + 3x^2 + x$.
4. $x^3 + 6x^2 + 3x - 10$.
5. $x^3 + 3x^2 - 5x - 15$.
6. $x^3 - 12x^2 + 27x + 40$.
7. $x^3 + 8x^2 + 17x + 10$.
8. $x^4 - 2x^2 + 1$.
9. $x^4 + x^3 + 8x + 8$.

Show without actual division

10. That $a^4 - b^4$ is divisible by $a - b$ and by $a + b$.
11. That $x^5 - 5x^4y + 11x^3y^2 - 14x^2y^3 + 9xy^4 - 2y^5$ is divisible by $x - y$.
12. That $a^5 - b^5$ is divisible by $a - b$, but not by $a + b$.
13. That $a^6 + b^6$ is not divisible by $a + b$ or $a - b$.

29. The sum or difference of the same two powers of two numbers.

The type form is

$$a^n \pm b^n.$$

By applying the factor theorem, we can determine whether $a + b$ or $a - b$ is a factor of $a^n \pm b^n$.

1. Divisibility by $a - b$.

Let n be either an even or an odd integer. If we substitute $a = b$ in $a^n - b^n$, we have $b^n - b^n = 0$; if we substitute $a = b$ in $a^n + b^n$ we have $b^n + b^n = 2b^n$.

Hence $a - b$ *is a factor of* $a^n - b^n$, *but not of* $a^n + b^n$.

2. Divisibility by $a + b$.

Let n be odd. Substituting $a = -b$ in $a^n - b^n$ we have $(-b)^n - b^n = -2b^n$; substituting in $a^n + b^n$ we have $(-b)^n + b^n = -b^n + b^n = 0$.

Therefore *if n is odd, $a + b$ is a factor of $a^n + b^n$, but not of $a^n - b^n$.*

Let n be even. Substituting $a = -b$ in $a^n - b^n$ we have $(-b)^n - b^n = b^n - b^n = 0$; substituting in $a^n + b^n$, we have $(-b)^n + b^n = b^n + b^n = 2b^n$.

Therefore *if n is even, $a + b$ is a factor of $a^n - b^n$,* but not of $a^n + b^n$.

Example 1. Factor $a^5 - b^5$. Find two factors only.

SOLUTION: $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

The second factor is found by division.

Example 2. Factor $a^7 + b^7$. Find two factors only.

SOLUTION: $a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6)$.

Example 3. Factor $a^{10} - b^{10}$. Find four factors.

SOLUTION: $a^{10} - b^{10} = (a^5 - b^5)(a^5 + b^5)$. Finish the solution.

Example 4. Factor $x^{10} + 243y^5$. Find two factors only.

SOLUTION:

$$\begin{aligned} x^{10} + 243y^5 &= (x^2)^5 + (3y)^5 \\ &= (x^2 + 3y)[(x^2)^4 - (x^2)^3 3y + (x^2)^2(3y)^2 - x^2(3y)^3 + (3y)^4] \\ &= (x^2 + 3y)(x^8 - 3x^6y + 9x^4y^2 - 27x^2y^3 + 81y^4). \end{aligned}$$

The above examples illustrate the following rules which are useful when we wish to find the quotient of $a^n \pm b^n$ divided by $a + b$ or by $a - b$.

When the divisor is $a - b$ all the terms of the quotient are positive.

When the divisor is $a + b$ the terms of the quotient are alternately positive and negative.

The number of terms in the quotient is n, if the divisor is a factor of the dividend.

The exponent of a in the first term is $n - 1$ and decreases by 1 in each succeeding term.

The exponent of b is 1 in the second term and increases by 1 in each succeeding term.

EXERCISES

Factor:

1. $x^3 + y^3$.
2. $x^7 - y^7$. Find two factors only.
3. $x^9 - y^9$. Find three factors only.
4. $x^6 + y^6$. Find two factors only.
5. $32a^5 - 1$. Find two factors only.
6. $x^{14} + y^7$. Find two factors only
7. $a^{10} - b^5$. Find two factors only.
8. $x^{12} - y^{12}$. Find six factors.
9. $128x^7 - y^{21}$. Find two factors only.
10. $243a^{10} + m^5$. Find two factors only.
11. Find the first four terms in the quotient $\frac{x^n - y^n}{x - y}$.
12. Find the first four terms in the quotient $\frac{x^n + y^n}{x + y}$.
13. Find the remainder when $x^5 - 7x^4 + x^3 - 5x^2 - x + 7$ is divided by $x - 1$; by $x + 1$.
14. Find the remainder when $4x^5 - 2x^4 - 7x^3 + x + 9$ is divided by $x - 2$; by $x + 3$.

30. Summary of factoring. The following suggestions will be found helpful in factoring.

I. Take out monomial factors, keeping in mind numerical factors.

II. After the monomial factors have been removed, if such factors occur, see if the expression to be factored can be classed under any of the following type forms of this chapter:

1. Binomials.
   (*a*) The difference of two squares, as $a^2 - b^2$.
   (*b*) The sum or difference of the same two powers, as $a^n \pm b^n$.
2. Trinomials.
   (*a*) Trinomial squares, as $a^2 \pm 2ab + b^2$.
   (*b*) Trinomials of the form $x^2 + (a + b)x + ab$.
   (*c*) The general quadratic trinomial, $ax^2 + bx + c$.
3. Polynomials of four or more terms.
   (*a*) Expressions to be factored by grouping terms.
   (*b*) Difference of two squares or expressions that can be reduced to that form.
   (*c*) Square of a polynomial.
   (*d*) Cube of a binomial.

III. See if a rearrangement of terms will bring the expression under any of the above forms.

IV. If the above methods fail, try the factor theorem.

V. Test each factor to see if it can be further factored.

VI. It is convenient to remember that $x^2 + y^2$, $x^2 \pm xy + y^2$, and $x^2 \pm xy - y^2$ are prime.

MISCELLANEOUS EXERCISES

Factor the following expressions:

1. $x^3 + 4x^2 + 4x$.
2. $x^2 + (a - b)x - ab$.
3. $16x^3 + 250y^6$.
4. $mx - nx + px$.
5. $9(2a - d)^2 - 4(3a - x)^2$.
6. $(4a - 3b)(7m - 2p) + (a + 4b)(7m - 2p)$.
7. $8a^2 - 21ab - 9b^2$.
8. $ax - bx + cx + ay - by + cy$.
9. $8m^3 - 36m^2q - 27q^3 + 54mq^2$.
10. $16a^4x^4 - 8a^2x^2 + 1$.

11. $m^2 + 4n^2 + 1 + 4mn - 2m - 4n$.

12. $a^6 + b^6$.

13. $m^5 - b^5$. Find two factors only.

14. $a^4 + b^4$.

15. $x^3 + x^5$.

16. $81a^2 - 16(2a - 3x)^2$.

17. $x^{2n} - y^{2n}$. Find two factors only.

18. $x^{4n} - y^{4n}$. Find three factors.

19. $x^2 - (c + 5)x + 5c$.

20. $n(x - y) - x + y$.

21. $abx^3 + 3abx^2 - abx - 3ab$.

22. $30x^2 + 13x - 77$.

23. $a^5 + 27a^2$.

24. $12ax^3y^4 - 36a^2bx^2y^2 + 27a^3b^2x$.

25. $x^3 + 5x^2 - 29x - 105$.

26. $x^2 + y^2 + 4 + 2xy - 4x - 4y$.

27. $a^4 + 4$.

HINT: Add and subtract $4a^2$.

28. $x^4 - 10x^2 + 9$.

29. $a^{2n} + 11a^n + 18$. Find two factors only.

30. $m^3 + n^3 + m + n$.

31. $k^{10} - k$. Find four factors.

32. $\frac{a^2}{x^2} - 1$.

33. $x^2 + x + \frac{1}{4}$.

34. $r^{2n+4} - r^2$.

35. $8m^3n^3 - 36m^2n^2x + 54mnx^2 - 27x^3$.

36. $x^3 + 8x^2 + x - 42$.

37. $12x^4 + 96x$.

38. $6x^2 + x - 2$.

39. $5x^2 + 2x - 3$.

40. $a^2 - 4ac + 4c^2 - x^2$.

41. $a^{2n} - 3a^{2n} + 3a^n - 1$.

42. $m^5 + 1$. Find two factors only.

43. $5^{3q} - 1$. Find two factors only.

44. $2^{4x} - 7^{2x}$. Find two factors only.

45. $x^3 - 3x + 2$.

46. $10^{6n} + 1000$. Find two factors only.

47. $a^4 + 4b^4$.

48. $3bmx + 2bm - 3anx - 2an$.

49. $b^3 - 4b^2 + 8$.

50. $a^2 - b^2 - 3ab(a + b)$.

51. $(m + n)^3 - 3(m + n)^2p + 3(m + n)p^2 - p^3$.

52. $m + 7 + (m + 7)^2$.

53. $a^6 + 19a^3 - 216$.

54. $c^2 + \frac{11}{30}c + \frac{1}{30}$.

55. $(x + y)^2 - x^2 + y^2$.

56. $3x^5 - 150x^3 + 147x$.

57. $x^4 + 2x^3 + 3x^2 + 2x + 1$.

58. $x^2 - 54x - 171$.

59. $2x^4 + 10x^3 - 252x^2$.

60. $x^{3n} + x^{2n} + x^n + 1$. Find two factors only.

61. $\frac{1}{9}x^2 + \frac{4}{3}x + 4$.

62. $p^2x^{2n} - 2pqx^ny^n + q^2y^{2n}$. Find two factors only.

63. $2^{3x} - 3^{3y}$. Find two factors only.

64. $125x^{15} + 75x^{10} + 15x^5 + 1$.

65. $56x^2 - 38x + 6$.

66. $15(k - 1)^2 - 2(k - 1) - 8$.

67. $(r - 2)^3 + 2(r - 2)^2$.

68. $x^7 + y^7$.

69. $r^9 - s^9$.

70. $1 - (4.5)^{4k}$. Find two factors only.

31. Solution of equations by factoring. The method of solving quadratic equations by factoring (Art. 84, First Course) can be extended to equations of higher degree than the second.

Example. Solve the equation $3x^3 + 16x^2 + x - 20 = 0$.

Factoring the left member, we have

$$(x - 1)(x + 5)(3x + 4) = 0.$$

We next set each of these factors equal to zero and solve for x and obtain $x = 1$, $x = -5$, and $x = -\frac{4}{3}$. These values check when substituted for x in the original equation.

EXERCISES

Solve the following equations and check:

1. $x^2 + 8x - 209 = 0$.
2. $6x^2 + 7x - 5 = 0$.
3. $x^3 + 4x^2 + x - 6 = 0$.
4. $10x^3 - 11x^2 - 6x = 0$.
5. $x^3 - 5x^2 - x + 5 = 0$.
6. $x^3 - 7x + 6 = 0$.
7. $x^3 - 8x = 0$.
8. $x^3 + x^2 = 0$.
9. $x^4 - 9x^2 + 8 = 0$.
10. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.
11. $x^2 - 5ax - 24a^2 = 0$.
12. $x^2 = 16$.
13. $x^3 - x^2 - x + 1 = 0$.
14. $2x^3 - 7x^2 + 7x = 2$.
15. $x^4 - 2a^2x^2 + a^4 = 0$.
16. $x^4 - 4x^3 = 12x^2$.
17. $3x^3 - x^2 - 3x + 1 = 0$.
18. $x^3 - 6x^2 + 11x - 6 = 0$.
19. $x^3 + 5x^2 - 2x = 10$.
20. $30x^4 + 13x^3 = 77x^2$.

32. Highest common factor. A number or expression which is a factor of two or more expressions, is called a **common factor** of those expressions.

The product of all the common prime factors of two or more expressions is called their **highest common factor** (H.C.F.).

Two expressions which have no common factor are said to be **prime to each other.**

The H.C.F. of two or more expressions can be found by resolving each of the expressions into its prime factors, and then finding the product of the common prime factors.

EXERCISES

Find the H.C.F. of each of the following sets of expressions:

1. ax^2, $2abx$, and $3a^2b^2$.
2. $12abc$, $28a^2bc^2x$, and $4a^4b^3cy$.
3. 60, 144, 84, and 156.
4. 52, 117, and 78.
5. $a^2 - b^2$ and $a^2 - 5ab + 4b^2$.
6. $x^2 + 2xy + y^2$, $x^2 + xy$, and $x^2 - 7xy - 8y^2$.
7. 240, 1260, and 3300.
8. 900, 1650, 3150, and 4850.
9. $12a^2 - 24ab$, $a^2 - 3b + 2b^2$, and $2a^2 - 3ab - 2b^2$.
10. $m^2 - 3m - 4$, $2m^3 - 7m^2 - 5m + 4$, and $m^3 - 4m^2 - m + 4$.
11. $x^3 - y^3$, $x^2 + 2xy + y^2$, and $x^4 - y^4$.
12. $r^2 - 6r + 9$, $r^2 + 5r - 24$, and $r^2 - 9r + 18$.
13. $x^3 + 3x^2y + 3xy^2 + y^3$, $x^3 + 2x^2y + xy^2$, and $x^2y + 2xy^2 + y^3$.
14. $4 - x^2$, and $x^2 - 4x + 4$.
15. $1 - x^3$, $x^2 + 13x - 14$, and $1 - x^2$.
16. $x^2 - (y + z)^2$, $(y - x)^2 - z^2$, and $y^2 - (x - z)^2$.
17. $ma + mb - mc$, $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$, and $(a + b)^2 - c^2$.
18. $x^3 - y^3$, $x^6 - y^6$, and $x^9 - y^9$.
19. $x^{2n} - x^ny^n$, $x^{2n} - 2x^ny^n + y^{2n}$, and $x^{2n} - y^{2n}$.
20. $(x^2 - 1)^2$, $x^4 - 1$, and $x^3 + x^2 - x - 1$.

33. Lowest common multiple. The **lowest common multiple** (L.C.M.) of two or more expressions has been defined to be the product of all of their different prime factors, each taken the greatest number of times that it occurs in any of the expressions. This definition gives the method of finding the L.C.M.

It may be noted that the L.C.M. of two or more expressions is the expression of lowest degree which contains each of the given expressions as a factor.

EXERCISES

Find the L.C.M. of each of the following sets of expressions:

1. $3ax^2$, a^2bx, and $2ab^2x$.
2. $6m^2n$, $-4mnx^3$, and $12mn^2x$.
3. 240, 225, and 500.
4. 160, 450, 600, and 840.
5. $x^2 + xy$, $x^3 + y^3$, and $x^2 - 3xy - 4y^2$.
6. $a^2 - 14a - 32$, $a^3 + 2a^2$, and $3a^4b$.
7. $x + 1$, $x + 2$, and $x^2 + 3x + 2$.
8. $a^2 + 7a + 10$, $a^3 + 6a^2 + 3a - 10$, and $a^2 + 4a - 5$.
9. $m^4 - 8m^2 + 16$, $5m^2 + 10m$, and $3m$.
10. $ax - ay + bx - by$, $a^2 + 2ab + b^2$, and $x^2 - xy$.
11. a^2, $a - 1$, $(a - 1)^2$, and $(a^2 - 1)^2$.
12. $2m^2 + 2m - 1$, $3m^3 - 4m + 1$, and $2m^3 - 3m + 1$.
13. $a(m - n)(m - p)$, $b(p - m)(m - q)$, and $c(q - m)(n - m)$.
14. $(x - 3)(x - 7)$, $(3 - x)(x - 2)$, and $(7 - x)(2 - x)$.
15. $a^3 + 8$, $8 - a^3$, and $a^2 - 4$.

CHAPTER IV

FRACTIONS

34. Algebraic fraction. An **algebraic fraction** is the indicated quotient of two expressions. Thus,

$$\frac{a}{b}$$

means a divided by b.

35. Reduction of fractions. The form of a fraction may be modified in various ways without changing the value of the fraction. For different purposes in the treatment of fractions, we make use of the following principles which have already been studied in the First Course:

I. *The value of a fraction is not changed by multiplying or dividing both the numerator and denominator by the same number.*

That is,
$$\frac{a}{b} = \frac{ax}{bx}.$$

II. *Changing the sign of either the numerator or the denominator of a fraction is equivalent to changing the sign of the fraction.*

That is,
$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}.$$

III. *Adding two fractions having a common denominator gives a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator.*

That is,
$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}.$$

Likewise,
$$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

IV. *The sum and the difference of two fractions are expressed*

$$by\ \frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd},\ and\ \frac{a}{b}-\frac{c}{d}=\frac{ad-bc}{bd},\ respectively.$$

We can reduce $\frac{a}{b}$ and $\frac{c}{d}$ to a common denominator, since by I,

$$\frac{a}{b}=\frac{ad}{bd},\ \text{and}\ \frac{c}{d}=\frac{bc}{bd}.$$

By III, we can complete the process.

V. *The product of two fractions is a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.*

That is, $$\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}.$$

VI. *To divide one fraction by another, invert the divisor and then multiply.*

That is, $$\frac{a}{b}\div\frac{c}{d}\ \text{or}\ \frac{\frac{a}{b}}{\frac{c}{d}}=\frac{ad}{bc}.$$

The **reciprocal** of a number is 1 divided by the number. Thus, the reciprocal of a is $\frac{1}{a}$; of $\frac{a}{b}$ is $\frac{b}{a}$; of 3 is $\frac{1}{3}$. Hence, to divide by a fraction, multiply by its reciprocal.

Propositions III and V, stated for two numbers in each case, are readily extended to three or more numbers.

36. Reduction to lowest terms. To reduce a fraction to its lowest terms separate the numerator and denominator into their prime factors and then cancel common factors by division. We know by Principle I (Art. 35) that this cancellation may be performed.

Reduce $\frac{x^3-8}{(x-2)^3}$ to its lowest terms.

SOLUTION:
$$\frac{x^3-8}{(x-2)^3} = \frac{(x-2)(x^2+2x+2)}{(x-2)^3}$$
$$= \frac{x^2+2x+2}{(x-2)^2}.$$

EXERCISES

Reduce to lowest terms:

1. $\frac{48}{60}$.
2. $\frac{51x}{85y}$.
3. $\frac{a^2}{a^2+ab}$.
4. $\frac{a^2+ab}{a^2-ab}$.
5. $\frac{a-1}{1-a}$.
6. $\frac{5a^2b^3c}{25a^3b^2c}$.
7. $\frac{x}{x-x^2}$.
8. $\frac{a^3-b^3}{a^2-b^2}$.
9. $\frac{a^3+b^3}{a^2+b^2}$.
10. $\frac{mv+nv-qv}{ny+my-qy}$.
11. $\frac{(m+n)^3}{m^3+n^3}$.
12. $\frac{x^{2n}-y^{2n}}{(x^n+y^n)^2}$.
13. $\frac{2a^2x^2-a^3x-6a^4}{x^3-3ax^2+2a^2x}$.
14. $\frac{x^2+y^2+z^2+2xy-2xz-2yz}{x^2+2xy+y^2-z^2}$.
15. $\frac{x^3+6x^2+11x+6}{x^3-7x-6}$.
16. $\frac{x+1}{x+1+(x+1)^2}$.
17. $\frac{(a-b)(c-d)(b-c)}{(a-c)(c-b)(a-b)}$.
18. $\frac{(x^4-y^4)(x^3-y^3)}{(x^3+y^3)(x^2+y^2)}$.

19. $\dfrac{x^3 - 8x^2 - 59x - 78}{x^2 - 11x - 26}$.

20. $\dfrac{x^{2n+3} - x^{2n}y^3}{x^{3n+1} - x^{3n}y}$.

21. $\dfrac{20a^3m^2 - 5a^5m^2n^4}{20a^2m - 20a^3mn^2 + 5a^4mn^4}$.

22. $\dfrac{(a^2 + b^2)^2}{a^4 - b^4}$.

23. $\dfrac{a^{2n}b^nc + a^nbc^{2n} + ab^{2n}c^n}{abc}$.

24. $\dfrac{(a+b)^3 + 4(a+b)^2 + (a+b)}{(a+b)^2}$.

37. Addition and subtraction. The method of addition and subtraction of fractions is given in Principles III and IV (Art. 35).

That is, we reduce the fractions to be added or subtracted to a common denominator, and then add or subtract the numerators.

Example. Add $\dfrac{1}{(x-2)(x-3)}$ and $\dfrac{2}{(x-3)(x-4)}$.

SOLUTION:

$$\frac{1}{(x-2)(x-3)} + \frac{2}{(x-3)(x-4)}$$

$$= \frac{x-4}{(x-2)(x-3)(x-4)} + \frac{2x-4}{(x-2)(x-3)(x-4)}$$

$$= \frac{3x-8}{(x-2)(x-3)(x-4)}.$$

EXERCISES

Perform the following additions and subtractions:

1. $\dfrac{a}{3} + \dfrac{2a}{3}$.

2. $\dfrac{1}{a} + \dfrac{1}{b}$.

3. $\dfrac{x}{4} + \dfrac{y}{5}$.

4. $\dfrac{x}{5} - \dfrac{x-1}{5}$.

5. $\frac{a}{x} - \frac{a+b}{2x}$.

6. $\frac{1}{am} + \frac{2}{bn}$.

7. $\frac{3x-2}{5} - \frac{5x-3}{2}$.

8. $1 + \frac{1}{b}$.

9. $\frac{7}{a+b} - \frac{2}{a} - \frac{5}{b}$.

10. $\frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3}$.

11. $\frac{x}{a+1} - \frac{x}{a-1}$.

12. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$.

13. $\frac{5}{3x-3} - \frac{8}{5x-15}$.

14. $a + b + \frac{a}{b} - 1$.

15. $\frac{5x-4}{x-2} + \frac{x^2-2x-17}{x^2-5x+6}$.

16. $\frac{4}{x+1} - \frac{x-2}{x^2+x} - \frac{3x}{x^2-1}$.

17. $\frac{1}{x+a} - \frac{2}{x+b} + \frac{3}{x^2+(a+b)x+ab}$.

18. $\frac{a}{b-c} + \frac{b}{c-a} - \frac{ab}{(a-c)(c-b)}$.

19. $\frac{8}{2m-3} + \frac{5}{3-2m} - \frac{3m-4}{2m^2-m-3}$.

20. $\frac{2a}{a-1} - \frac{a+1}{a^2+1} - \frac{3a}{a+1} + \frac{a^2-a+1}{a^2-1}$.

21. $\frac{a}{a-b} + \frac{b}{b-a} + 1$.

22. $\frac{1}{(x-y)(y-z)} + \frac{1}{(x-z)(z-y)} - \frac{1}{(y-x)(z-x)}$.

23. $\frac{x-y}{x^2-5xy+6y^2} - \frac{x-2y}{3y^2-4xy+x^2}$.

24. $\frac{x+a}{x^2-(b+c)x+bc} - \frac{x+b}{x^2-(a+c)x+ac} + \frac{x+c}{x^2-(a+b)x+ab}$.

25. $\frac{x+1}{(x-1)^3} + \frac{x}{(x-1)^2} + \frac{1}{x-1} + 1$.

26. $\dfrac{a^n + b^n}{a^n - b^n} + \dfrac{a^n - b^n}{a^n + b^n}$.

27. $\dfrac{3a}{a^2 + a - 20} - \dfrac{a - 1}{a^2 - 15a + 44} + \dfrac{2}{a^2 - 6a - 55}$.

28. $\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{1}{a + b} + \dfrac{1}{a - b}$.

29. $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4} + \dfrac{1}{x^5}$.

30. $\dfrac{1}{x^3 + y^3} - \dfrac{x^3}{x^6 - y^6} - \dfrac{y^3}{(x^3 + y^3)^2}$.

31. $1 - y - y^2 - \dfrac{y - y^2}{1 - y^4}$.

32. By how much is the value of the fraction $\frac{8}{13}$ changed by adding 6 to both terms? By subtracting 6 from both terms? Answer the same question for $\frac{13}{8}$.

33. Find the value of the fraction $\dfrac{1 + x}{2 + x}$ (*a*) when x has the values 1, 10, 100, 1000. (*b*) When x has the values .1, .01, .001. Arrange the results in order of magnitude in each case.

38. Multiplication and division. The rules for multiplication and division are given in Principles V and VI (Art. 35).

Example 1. Find $\dfrac{x^2 - 2ax + a^2}{x^2 - a^2} \cdot \dfrac{x + a}{x^2 + a^2}$.

SOLUTION: $\dfrac{\cancel{x - a}}{\dfrac{\cancel{x^2 - 2ax + a^2}}{\cancel{x^2 - a^2}}} \cdot \dfrac{\cancel{x + a}}{x^2 + a^2} = \dfrac{x - a}{x^2 + a^2}$.

Example 2. Divide $\dfrac{x^2 - 2ax + a^2}{x^2 - a^2}$ by $\dfrac{x - a}{x^2 + a^2}$.

SOLUTION: $\dfrac{x^2 - 2ax + a^2}{x^2 - a^2} \div \dfrac{x - a}{x^2 + a^2} = \dfrac{\cancel{x - a}}{\dfrac{\cancel{x^2 - 2ax + a^2}}{\cancel{x^2 - a^2}}} \cdot \dfrac{x^2 + a^2}{x - a} = \dfrac{x^2 + a^2}{x + a}$.

EXERCISES

Perform the following multiplications and divisions and simplify the results:

1. $a \cdot \frac{b}{a}$.
2. $8 \cdot \frac{7}{24}$.
3. $\frac{6x}{5y} \cdot \frac{2y}{3}$.
4. $8k \cdot 1\frac{1}{2}$.
5. $\frac{8ax}{5by} \cdot \frac{3ay}{4bx}$.
6. $\frac{7}{8} \cdot \frac{12}{7} \cdot \frac{40}{51}$.
7. $\frac{5x}{4} \cdot \frac{4x}{5} \cdot \frac{1}{x}$.
8. $6b \div \frac{2}{3}$.
9. $7c \div \frac{7}{5}$.
10. $\frac{5ab}{6xy} \div \left(\frac{a}{2x}\right)^2$.
11. $\frac{6ab}{a-b} \div \frac{8ax}{a+b}$.
12. $xyz\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$.
13. $\left(\frac{a}{b} + \frac{b}{c}\right) \div \frac{c}{ab}$.
14. $\frac{a}{a-b} \cdot \frac{b}{b-a}$.
15. $\frac{a^2 - ab}{x^2 - xy} \cdot \frac{x^2 + xy}{a^2 + ab}$.
16. $\left(a + \frac{ab}{a-b}\right)\left(b - \frac{ab}{a+b}\right)$.
17. $\frac{m+1}{n} \div \frac{m^2 - 1}{2n^2}$.
18. $\left(1 + \frac{1}{n}\right) \div \left(2 + \frac{2}{m}\right)$.
19. $\frac{(a-x)^3}{x^3 + y^3} \cdot \frac{x^2 - xy + y^2}{a^2 - 2ax + x^2}$.
20. $\frac{x^2 + x - 2}{x^2 - 7x} \cdot \frac{x^2 - 13x + 42}{x^2 + 2x}$.
21. $\frac{m^2 + 2mn}{m^2 + 4n^2} \div \frac{m^2 - 4n^2}{mn - 2n^2}$.
22. $\left(\frac{x^2}{a^2} - \frac{x}{a} + 1\right) \div \left(\frac{x^2}{a^2} + \frac{x}{a} + 1\right)$.
23. $\frac{1}{a^2}\left(\frac{a}{b} - \frac{b}{a}\right) \cdot \left(\frac{a^3}{a-b}\right)^2 \div \left(1 - \frac{b}{a}\right)$.

24. $(3x^2 - 7xy + 4y^2) \cdot \left(\frac{2x}{x^2 + 3xy - 10y^2}\right) \div \left(\frac{3x - y}{y}\right)^2$.

25. $\left(\frac{1}{a} - \frac{1}{b}\right)\left(1 - \frac{2b}{a} + \frac{b^2}{a^2}\right) \div (a - b)^2$.

26. $(a^3 + 3a^2 - 6a - 8) \div \frac{a^2 + 2a - 8}{a + 1}$.

27. $\left(\frac{1}{a} - \frac{1}{b}\right) \div \left(\frac{1}{a} + \frac{1}{b}\right) \cdot \frac{(a + b)^2}{(a - b)^2}$.

28. $\frac{x^2 + y^2}{x^3 - y^3} \cdot \frac{x^4 + y^4}{x^4 - y^4} \div \frac{x^8 - y^8}{x^6 - y^6}$.

29. $\frac{a^2m^2x - a^2b^4x^3}{(m - b^2x)^3} \cdot \frac{m^3 - b^6x^3}{abm^2 + 2ab^3mx + ab^5x^2}$.

30. $\frac{z^3 - 6z^2 + 36z}{z^2 - 49} \div \frac{z^4 + 216z}{z^2 - z - 42}$.

39. Complex fractions. A **complex fraction** is a fraction that contains a fraction either in its numerator, or in its denominator, or in both. Complex fractions are simplified by the rules for division of fractions.

One of the most useful methods of simplifying complex fractions consists in multiplying both numerator and denominator of the complex fraction by the L.C.M. of the denominators of the simple fractions that make up the terms.

Example 1.
$$\frac{\frac{2}{3} - \frac{1}{2}}{\frac{3}{4} - \frac{5}{12}} = \frac{12\left(\frac{2}{3} - \frac{1}{2}\right)}{12\left(\frac{3}{4} - \frac{5}{12}\right)} = \frac{8 - 6}{9 - 5} = \frac{2}{4} = \frac{1}{2}.$$

Example 2.
$$\frac{\frac{1}{b} + \frac{1}{a + b}}{\frac{1}{a} - \frac{1}{a - b}} = \frac{ab(a^2 - b^2)\left[\frac{1}{b} + \frac{1}{a + b}\right]}{ab(a^2 - b^2)\left[\frac{1}{a} - \frac{1}{a - b}\right]}$$
$$= \frac{a(a^2 - b^2) + ab(a - b)}{b(a^2 - b^2) - ab(a + b)}$$
$$= -\frac{a(a^2 - 2b^2 + ab)}{b^2(a + b)}.$$

EXERCISES

Simplify the following fractions:

1. $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{5}{6}}$.

2. $\dfrac{2-\frac{3}{4}}{1+\frac{1}{8}}$.

3. $\dfrac{a-\frac{1}{b}}{b-\frac{1}{a}}$.

4. $\dfrac{\frac{3}{5}+\frac{5}{6}}{\frac{14}{15}-\frac{1}{2}}$.

5. $\dfrac{\frac{x}{x-1}-1}{1+\frac{x}{1-x}}$.

6. $\dfrac{\frac{a}{b}+\frac{c}{d}}{\frac{m}{n}-\frac{p}{q}}$.

7. $\dfrac{x-y+\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}+1}$.

8. $\dfrac{1}{2+\frac{3}{4+\frac{5}{6}}}$.

9. $\dfrac{a}{b+\frac{c}{d+\frac{e}{f}}}$.

10. $\left(\dfrac{a^2+ax}{2x}\right)\left(\dfrac{(a+x)^2}{4ax}-1\right)$.

11. $\dfrac{1+\frac{1}{a^2}+\frac{1}{a^4}}{1+\frac{1}{a}+\frac{1}{a^2}}$.

12. $\dfrac{\frac{a+b}{ab}\left(\frac{1}{a}-\frac{1}{b}\right)-\frac{b+c}{bc}\left(-\frac{1}{c}-\frac{1}{b}\right)}{\frac{a+c}{ac}\left(\frac{1}{a}-\frac{1}{c}\right)}$.

13. $\dfrac{\left(\frac{a+b}{a^2+ab+b^2}\right)\left(a^2+\frac{b^4}{a^2+b^2}\right)}{1\div\left(\frac{a}{a+b}+\frac{b}{a-b}\right)}$.

14. $x+1+\cfrac{x}{x+2-\cfrac{x+1}{x+\cfrac{1}{x+2}}}$.

15. $\cfrac{m-n}{m-n+\cfrac{1}{m+n+\cfrac{1}{m-n}}}$.

16. $\cfrac{\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}}{\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}}\cdot\left[\frac{(a+b+c)^2}{ab+bc+ca}-2\right]$.

17. $\cfrac{\frac{1}{p-q}-\frac{1}{p-r}}{\cfrac{q^2}{p^2-p(q+r)+qr}}$.

18. $\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\frac{g}{h}}}}$.

19. $\cfrac{1-x^2}{1-\cfrac{x}{1-x+\cfrac{x^2}{1-x}}}$.

20. $\cfrac{\frac{1}{2}}{\frac{1}{4}+\cfrac{\frac{1}{6}}{\frac{1}{8}+\cfrac{\frac{1}{10}}{\frac{1}{12}+\frac{1}{14}}}}$.

40. Equations involving fractions. In solving an equation that involves fractions, it is usually convenient to **clear of fractions** by multiplying each member by the L.C.D. of the fractions.

If the unknown occurs in any denominator, multiplying by the L.C.D. may or may not introduce new roots that do not satisfy the equation to be solved.

Example 1. Solve $\dfrac{5}{x-1}+\dfrac{3}{x-3}=2.$

SOLUTION: $$\frac{5}{x-1}+\frac{3}{x-3}=2. \tag{1}$$

(1) $\cdot$ $(x-1)(x-3)$ gives $5(x-3)+3(x-1)=2(x-1)(x-3).$ (2)

Simplifying (2), $$2x^2-16x+24=0. \tag{3}$$

$$2(x-6)(x-2)=0. \tag{4}$$

Hence, $x=6,$

or $x=2.$

The roots of (2) are 6 and 2, and both satisfy (1).

It is important to note that if we should multiply (1) by a common denominator other than the L.C.D., roots may be introduced which do not satisfy (1). To illustrate, if the members of (1) are multiplied by $(x-1)(x-3)(x-4)$, the resulting equation,

$$5(x-3)(x-4)+3(x-1)(x-4)=2(x-1)(x-3)(x-4),$$

has the root $x=4$ that does not satisfy (1).

Example 2. Solve $\dfrac{x-1}{x^2-1}=1.$

SOLUTION: $$\frac{x-1}{x^2-1}=1. \tag{1}$$

(1) $\cdot$ (x^2-1) gives $$x-1=x^2-1, \tag{2}$$

or $$x^2-x=0.$$

Hence, $$x=0 \text{ or } x=1. \tag{3}$$

The roots of (2) are 0 and 1. Now $x=0$ satisfies (1), but $x=1$ does not satisfy (1) since the left hand member has no meaning when $x=1$. Hence, the root $x=1$ is introduced in clearing of fractions.

The introduction of roots in clearing of fractions when an unknown occurs in the denominator should make clear the importance of checking each solution by substitution in the equation to be solved.

EXERCISES

Solve the following equations and check the results:

1. $\dfrac{3}{x-2}+\dfrac{2x}{x-3}=2.$

2. $\dfrac{2}{x-3}=\dfrac{3}{x-2}.$

3. $\dfrac{10}{x}+\dfrac{4}{9}=\dfrac{9}{x}+\dfrac{1}{2}.$

4. $\frac{2}{3}(7x-10)-\frac{1}{2}(50-x)=20.$

5. $\dfrac{x-9}{x-5}+\dfrac{x-5}{x-8}=2.$

6. $\dfrac{3}{x-7}+\dfrac{1}{x-9}=\dfrac{4}{x-8}.$

7. $\dfrac{5(2x^2+3)}{2x+1}-\dfrac{7x-5}{2x-5}=5x-6.$

8. $\dfrac{5}{x-6}+\dfrac{4}{x-9}=\dfrac{8}{x-7}+\dfrac{1}{x-10}.$

9. $\dfrac{x-2}{2x+1}+\dfrac{x-1}{3(x-3)}=\dfrac{5}{6}.$

10. $\dfrac{x-3}{2}+\dfrac{1}{x+4}=\dfrac{4x-1}{8}.$

11. $\dfrac{21}{x}-\dfrac{10}{x-2}-\dfrac{11}{x-3}=0.$

12. $\dfrac{x+11}{2x+3}=\dfrac{2x+1}{4x+5}.$

13. $\dfrac{5+x}{3-x}-\dfrac{8-3x}{x}=2.$

14. $3-\dfrac{1}{\frac{1}{3}+\frac{1}{x}}=\dfrac{1}{3}.$

15. $\dfrac{x}{.5+x}-.5=.5+\dfrac{.5-x}{.5+x}.$

16. $a - \frac{b}{x} = c.$

17. $\frac{x}{a} - b = \frac{x}{b} - a.$

18. $\frac{m+x}{m-x} = \frac{m+n}{m-n}.$

19. $\frac{x-p}{p} + \frac{x-q}{q} + \frac{x-r}{r} = -3.$

20. $1 = \frac{m-ab}{bx} - \frac{n-bk}{kx}.$

21. $\frac{n-x}{m+x} + \frac{p-x}{m-x} = \frac{m(p-2x)}{m^2-x^2}.$

22. $\frac{x}{rs} - \frac{1}{r+s} = (r+s)x - rs.$

23. $\frac{x-.5}{.2x+1} = \frac{3x}{.6x+3.5}.$

24. $\frac{5x-.4}{.3} + \frac{1.3-3x}{2} = \frac{1.8-8x}{1.2}.$

25. $\frac{2}{.3x} = \frac{8}{.5x-1} - \frac{7}{75(.01x+.2)}.$

EXERCISES AND PROBLEMS

1. What part of a piece of work can A do in 1 day if he can do all of the work (a) in 5 days; (b) in 17 days; (c) in $3\frac{1}{3}$ days; (d) in $5\frac{2}{7}$ days; (e) in m days; (f) in $m+n$ days? Tell in each case what part he can do in 3 days; in d days.

2. A can do a piece of work in m days, and B can do the work in n days. What part can each do in 1 day? What part can both together do in 1 day? How many days will be required by A and B working together to do the work?

3. The marked price of a bill of goods is \$600. It is discounted 7%. Find the discount and the selling price.

4. Find the discount and the selling price if the marked price is m dollars and the rate of discount r%.

5. Find the marked price and selling price if the discount is \$40 and the rate of discount 2%.

6. Find the marked price and selling price if the discount is d dollars and the rate of discount r%.

7. Find the marked price and the discount if the selling price is s dollars and the rate of discount r%.

8. In a division the dividend, divisor, quotient, and remainder are represented by D, d, q, and r, respectively. Express D in terms of d, q, and r. Express q in terms of D, d, and r.

9. If a oranges cost c cents, what is the cost of b oranges?

10. What is the average selling price for a horse if m horses are sold at x dollars each, and n horses are sold at y dollars each?

11. What is the average selling price if a articles are sold at p cents each, b at q cents each, and c at r cents each?

12. If a train goes m miles in x minutes, how far will it go in 1 hour?

13. At c cents a pound how many pounds of butter can be bought for d dollars?

14. At $\frac{a}{b}$ dollars per yard how many yards of muslin can be bought for \$7? For a dollars? For b dollars?

15. What is the gain per cent on the cost if an article is bought for \$120 and sold for \$150? If bought for x dollars and sold for y dollars?

16. What number added to both the numerator and the denominator of $\frac{6}{11}$ gives a fraction equal to $\frac{3}{4}$?

17. What number added to both the numerator and the denominator of $\frac{a}{b}$ gives a result equal to $\frac{c}{d}$?

18. If for the sum of the fractions $\frac{3}{5}$ and $\frac{7}{8}$ we take the sum of the numerators divided by the sum of the denominators, by how much does the result differ from the true result? Answer the same question for the fractions $\frac{a}{b}$ and $\frac{c}{d}$.

19. Divide the number n into two parts such that the first part equals $\frac{2}{3}$ of the second.

20. State the rules of arithmetic for multiplying and dividing a fraction by an integer. Apply these rules to multiplying and dividing $\frac{a}{b}$ by c.

REVIEW EXERCISES AND PROBLEMS ON CHAPTERS I–IV

Insert expressions in the parentheses so as to form identities.

1. $x + y - z = x + (\quad)$.

2. $x - y - z = x - (\quad)$.

3. $x - y + z = x - (\quad)$.

4. $x + 6y - 6 = x - 6(\quad)$.

5. $a - 25b + 10 = a - 5(\quad)$.

6. $ax - ay + bx - by = (a + b)(\quad)$.

7. How do you check the solution of an equation?

Solve the following for x and check the solutions:

8. $1 - \frac{2x}{3} + \frac{3x}{4} = 0$.

9. $1.4x - 1.1x = 0.3$.

10. $1.7(x - 2) - 0.3(2x + 1) = 0.8$.

11. $\frac{5(2 - x)}{6} - \frac{2(x - \frac{1}{4})}{4} = 0$.

12. $\frac{2x}{3} + \frac{3x + 5}{2} = 22$.

13. $\frac{1}{2x + 1} + \frac{2}{2x + 1} = 1$.

14. Solve $x - \left(\frac{1 - x}{4} - \frac{1 + x}{2}\right) = \frac{x + 3}{2}$. (DARTMOUTH *)

15. Remove the parentheses from

$$3a - \{3a - [3a - (3a - \overline{3a - 3a}) - 3a] - 3a\} - 3a,$$

and simplify the result. (DARTMOUTH)

16. Find the value of

$$a - \{5b - [a - (3c - 3b) + 2c - 3(a - 2b - c)]\},$$

where $a = -3$, $b = 4$, $c = -5$. (YALE)

* Institution that gave the question in an entrance examination.

17. Find the product: $(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16})$. (ILLINOIS)

18. Find the greatest common factor and the least common multiple of the three expressions:

$$a^4+a^2x^2+x^4,\ a^2+ax+x^2,\ a^2-ax+x^2. \qquad \text{(HARVARD)}$$

19. Simplify $\dfrac{\dfrac{a^2+b^2}{a}-b}{\dfrac{1}{a}-\dfrac{1}{b}} \div \dfrac{a^3+b^3}{a^3-b^3}$. (YALE)

20. Divide $6x^{3n}-25x^{2n}+27x^n-5$ by $2x^n-5$. (ILLINOIS)

Express the following in as few terms as possible:

21. $4.8x^2-2x^2+3.5y^2-2.4xy-6y^2+2x^2-(6.4xy-y^2)$.

22. $6.4x^2-[1.6x^2+(7x-1+1.4x^2)-3+6x]$.

23. A person who possesses \$15,000 employs part of the money in building a house. He invests one third of the money that remains at 6 per cent, and the other two thirds at 9 per cent, and from these two investments he obtains an annual income of \$500. What is the cost of the house? (MASS. INSTITUTE)

24. The admission to an entertainment is 50 cents for adults and 35 cents for children. If the proceeds from the sale of 100 tickets amounted to \$39.50, how many tickets of each kind were sold? (ILLINOIS)

25. Divide x increased by 3 by $3x$ decreased by 4. Divide the result by $3x^2$ less the binomial $3x-7$, and call the result A. Form the same expression where $x-1$ takes the place of x and call it B. Divide A by B and reduce to a simple fraction. (PRINCETON)

Factor the following:

26. $9a^4-16$.

27. $9a^4-4$.

28. $(x-y)^2-(x+y)^2$.

29. x^4+4x^2+4.

30. $35p^2+4pq-15q^2$.

31. x^4-16x^2+64.

32. $x(x+1)(x+3)+x^2+4x+3$.

33. x^4-1.

34. x^4+1.

35. x^8-1.

36. $x^3+6x^2+11x+6$.

37. $x^3-6x^2+11x-6$.

38. m^6-n^6.

39. m^8-n^8.

Find the values of expressions in Exercises 40–43 for $a = 5$, $x = 8$, $y = 7$, in the shortest way possible.

40. $\dfrac{\dfrac{x^2+a^2}{x^2-a^2}+\dfrac{x^2-a^2}{x^2+a^2}}{\dfrac{x^2+a^2}{x^2-a^2}-\dfrac{x^2-a^2}{x^2+a^2}}$.

41. $\dfrac{\left(\dfrac{x^2+y^2}{y}-x\right)\left(\dfrac{x^2-y^2}{x^3+y^3}\right)}{\dfrac{1}{y}-\dfrac{1}{x}}$.

42. $\dfrac{1}{x(x-1)(x-2)(x-3)}-\dfrac{1}{x(x+1)(x-1)(x-2)}$.

43. $\dfrac{1}{x(x+1)(x+2)(x+3)}-\dfrac{1}{(x+1)(x+2)(x+3)(x+4)}$.

44. Reduce to a single fraction

$$\frac{1}{x+3}+\frac{7}{x^2-x-12}+\frac{x+3}{x-4}.$$ (OHIO STATE)

45. Simplify: $\dfrac{1-x^2}{1+y}\cdot\dfrac{1-y^2}{x+x^2}\left(1+\dfrac{x}{1-x}\right)$. (OHIO STATE)

46. Find in the shortest way the numerical value of

$$(x^2-xy+y^2)(x+y),$$

if $x = 24$ and $y = 12$.

47. A train running 30 miles an hour requires 21 minutes longer to go a certain distance than does a train running 36 miles an hour. How great is the distance? (CORNELL)

48. Factor, and find the highest common factor and the lowest common multiple of the expressions $2x^4-x^3-x^2$, $2x^2+x-3$, x^3-x^2-x+1.

49. Simplify: $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4xy}{x^2-y^2}$. (OHIO STATE)

50. A bicyclist averaging 12 miles per hour is a half mile ahead of an automobile running 40 miles per hour. How many minutes before the automobile dashes past him? (STANFORD)

51. Given the equation $5x-10=3x-2$. What extraneous root is introduced into the equation by multiplying each member by $x-3$?

52. Verify that 2 is a root of $3x(x-2)=x^2-4$. Is 2 a root of the equation obtained by dividing each member of the given equation by $x-2$?

53. The difference of the squares of two consecutive integers is 41. What are the integers?

54. Concrete is to be made of a mixture of cement, sand, and gravel. The weights of sand and gravel used are equal, but the weight of the cement used is $\frac{1}{8}$ that of the sand. How many pounds of each are required to make 7800 pounds of concrete?

55. The formula for converting a temperature reading of F degrees Fahrenheit into its equivalent temperature of C degrees Centigrade is

$$C = \tfrac{5}{9}(F - 32).$$

Express F in terms of C, and compute F for $C = 25$ and for $C = 42$.

56. The numerator of a fraction is 7 less than the denominator; if 4 is subtracted from the numerator and 1 is added to the denominator, the resulting fraction equals $\frac{1}{3}$. Find the fraction.

57. Simplify the fraction $\dfrac{1}{a + \dfrac{1}{1 + \dfrac{a+1}{3-a}}}$. (NEBRASKA)

58. A train runs 100 miles in a certain time. If it were to run 5 miles an hour slower it would run 20 miles less in the same time. What is the rate at which the train runs? (MISSOURI)

59. Divide 54 into two parts such that twice the smaller shall exceed 29 as much as 143 exceeds four times the greater. (WISCONSIN)

60. Solve the equation

$$\frac{x+b}{2a} + \frac{2a}{x-b} = \frac{x}{2a} - \frac{2a}{b}\left(1 - \frac{2a-b}{x-b}\right).$$ (HARVARD)

61. Reduce the expression $\dfrac{b+a}{b-a} - 2\left(\dfrac{b}{a} - \dfrac{b}{a-b}\right)$, to a single fraction in its lowest terms. (HARVARD)

62. Which takes the lesser force to lift a weight of 100 pounds with a crowbar 6 feet long; (*a*) when the weight is at the end of the bar and the fulcrum one foot from the end; (*b*) when the fulcrum is at the end of the bar and the weight one foot from the end. Estimate your answer, then calculate the actual difference.

63. The distance from San Francisco to Los Angeles is 475 miles. A train running 26 miles per hour leaves San Francisco for Los Angeles at the same time that a train running 31 miles per hour leaves Los Angeles for San Francisco. In how many hours will they meet? (CALIFORNIA)

CHAPTER V

FUNCTIONS

41. Constants and variables. A **constant** is a symbol which represents the same number throughout a discussion.

Thus, in the formula for the volume V of a cylinder of radius r and height h,

$$V = \pi r^2 h,$$

the symbol π is a constant, whatever values r and h may have. In problems dealing with falling bodies the space s through which a body falls in time t is given by the formula

$$s = \tfrac{1}{2} g t^2,$$

where g is a constant.

The Arabic numerals are, of course, the most common examples of constants.

A **variable** is a symbol which may represent different numbers in the discussion or problem into which it enters.

Thus, in the foregoing illustrations, V, r, h, s, t are variables. Many mathematical expressions contain both variables and constants. Except in certain geometrical and physical formulas it is customary to use the letters a, b, $c \cdots$ from the beginning of the alphabet to denote constants and the letters $\cdots x$, y, z at the end of the alphabet, to denote variables.

Exercise. In Fig. 4, let the point P move around the circle, while the points P_1, O, and Q are fixed. What lines in the figure represent variables and what represent constants?

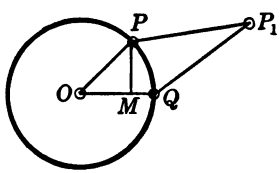

FIG. 4

42. Definition of a function. Many discussions and problems of algebra involve two variables which are so related that

when a value of one is given, a corresponding value of the other can be found. Throughout the course in algebra, we have had many problems in which two or more such related variables occur. In the equation $2x - 3y = 4$, if a value be given to x, the corresponding value of y can be found. Thus, if $x = 0$, $y = -\frac{4}{3}$; if $x = 1$, $y = -\frac{2}{3}$, and so on.

In evaluating the expression $x^2 + 3x - 1$, we find that $x^2 + 3x - 1 = -1$ when $x = 0$, $x^2 + 3x - 1 = 3$ when $x = 1$, and so on. Fixing the value of x in the first illustration fixes the value of y; in the second illustration fixing the value of x fixes the value of $x^2 + 3x - 1$.

DEFINITION. If two variables are so related that when a value of one is given, a corresponding value of the other is determined, the second variable is called a **function** of the first.

Thus, in the equation $2x - 3y = 4$, y is a function of x. The expression $x^2 + 3x - 1$, and in general any expression containing x, is a function of x. We may therefore and shall speak of a "function of x," instead of an "expression involving the variable x."

43. Functional notation. If we refer to the same function of x a number of times in a discussion, it is convenient to have some suggestive abbreviation to represent it. It has become customary to represent a function of x by the symbol $f(x)$ which is read "f function of x." If another function occurs in the same problem it can be represented by $F(x)$, which is then read "F major function of x," while the $f(x)$ is read "f minor function of x."

If in a discussion $f(x)$ is used as an abbreviation for the expression $x^4 + 3x^3 + 2x - 3$, then $f(a)$ means the result when a replaces x. Thus,

$$f(a) = a^4 + 3a^3 + 2a - 3,$$

$$f(h) = h^4 + 3h^3 + 2h - 3,$$

and

$$f(2) = 2^4 + 3 \cdot 2^3 + 2 \cdot 2 - 3.$$

Similarly, if $F(x) = \dfrac{x^2 - 2x + 4}{x + 1}$,

then $F(a) = \dfrac{a^2 - 2a + 4}{a + 1}$,

$F(h) = \dfrac{h^2 - 2h + 4}{h + 1}$,

and $F(2) = \dfrac{2^2 - 2 \cdot 2 + 4}{2 + 1}$.

These illustrations bring out an important point in this notation, viz.: If the x in the parentheses of the symbol of $f(x)$ is replaced by any other number, the new symbol means the value of $f(x)$ when the number is put in place of x.

44. Evaluation of functions. To evaluate a function of x is to put a given number in place of x and to calculate the corresponding value of the function.

Thus, if $f(x) = x^4 + 3x^3 + 2x - 3$, we evaluate it for $x = 2$ by replacing x by 2 and reducing. Thus,

$$f(2) = 2^4 + 3 \cdot 2^3 + 2 \cdot 2 - 3 = 41.$$

For $x = 3$, we find $f(3) = 3^4 + 3 \cdot 3^3 + 2 \cdot 3 - 3 = 165$.

EXERCISES

1. If $f(x) = 2x - 3$, find $f(1), f(0), f(2), f(-1)$.
2. If $f(x) = 3x + 2$, find $f(0), f(4), f(-4), f(a)$.
3. If $f(x) = x^2$, find $f(0), f(\frac{1}{2}), f(\frac{3}{4}), f(m)$.
4. If $f(x) = 2x^2 - 1$, find $f(\frac{1}{4}), f(7), f(11), f(p)$.
5. If $f(t) = 16t^2$, find $f(1), f(2), f(x), f(10)$.
6. If $f(y) = y^2 + y + 1$, find $f(2), f(2.3), f(.01), f(z)$.
7. If $f(s) = \dfrac{s - 1}{s + 1}$, find $f(3), f(9), f(1.1), f(t)$.
8. If $f(n) = \dfrac{n^2 + n + 1}{n^2 - n - 1}$, find $f(1), f(\frac{1}{2}), f(x), f(r)$.

9. If $f(x) = x^2 - 4x + 3$ and $F(x) = x^3 - 1$, find $f(2)$ and $F(2)$; $f(6)$ and $F(6)$.

10. If $f(x) = x^3 + x$ and $F(x) = 2x^2 - 4x - 5$, find the quotients $\frac{f(1)}{F(1)}$ and $\frac{f(2)}{F(3)}$.

45. Formulas frequently used. Most of the formulas of mensuration learned in arithmetic and geometry express variables as functions of other variables. Thus, for the area A of a circle of radius r, we have the formula

$$A = \pi r^2,$$

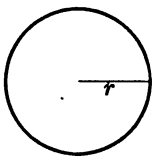

FIG. 5

which expresses the area as a function of the radius, so that for a given r the area A can be found.

The following list of common formulas will be found useful for reference.

1. Area A of a rectangle of sides a and b.

$$A = ab.$$

2. Area A of a parallelogram of base b and altitude h.

$A = bh$. (Fig. 6.)

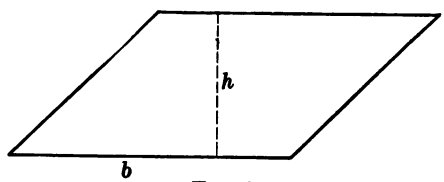

FIG. 6

3. Area A of a triangle of base b and altitude h.

$$A = \tfrac{1}{2}bh. \quad \text{(Fig. 7.)}$$

4. Area A of a triangle in terms of its sides, a, b and c.

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{a+b+c}{2}$. (Fig. 8.)

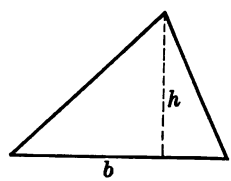

FIG. 7

5. Area A of a circle of radius r and circumference C.

$$A = \tfrac{1}{2}Cr.$$

6. Circumference C of a circle of diameter d, or of radius r.

$C = \pi d$, or $C = 2\pi r$. $\pi = 3.1416$.

A useful value for practical purposes is

$$\pi = \frac{22}{7} = 3.14 +.$$

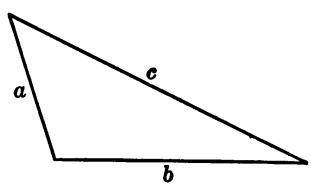

FIG. 8

7. Area A of a circle of diameter d or radius r.

$$A = \tfrac{1}{4}\pi d^2 \text{ or } A = \pi r^2.$$

8. Area A of a trapezoid of bases b_1 and b_2 and altitude h. (Fig. 9.)

$$A = \tfrac{1}{2}(b_1 + b_2)h.$$

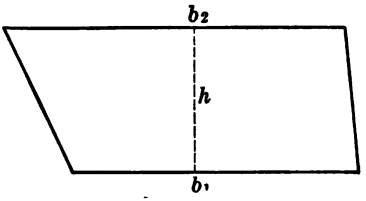

FIG. 9

9. Area A of a regular hexagon of side a. (Fig. 10.)

$$A = \frac{3\sqrt{3}}{2}a^2.$$

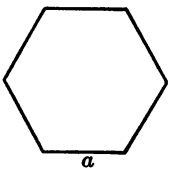

FIG. 10

10. Length c of the hypotenuse of a right triangle of sides a and b. (Fig. 11.)

$$c = \sqrt{a^2 + b^2}.$$

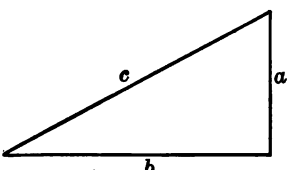

FIG. 11

11. Volume V of a cube of edge a. (Fig. 12.)

$$V = a^3.$$

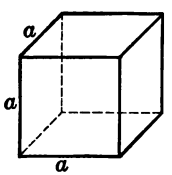

FIG. 12

12. Volume V of a rectangular solid of length l, width w and altitude h. (Fig. 13.)

$$V = lwh.$$

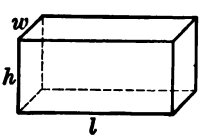

FIG. 13

13. Volume V of a cylinder of altitude h and radius of base r. (Fig. 14.)

$$V = \pi r^2 h.$$

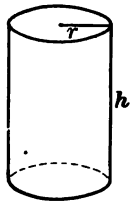

FIG. 14

14. Volume V of a pyramid of altitude h and area of base B. (Fig. 15.)

$$V = \tfrac{1}{3}Bh.$$

15. Volume V of a cone of altitude h and radius of base r. (Fig. 16.)

$$V = \tfrac{1}{3}\pi r^2 h.$$

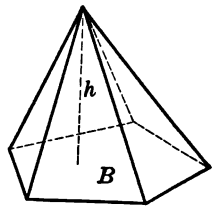

FIG. 15

16. Surface S of a sphere of radius r, or diameter d.

$$S = 4\pi r^2, \text{ or } S = \pi d^2.$$

17. Volume V of a sphere of radius r, or diameter d.

$$V = \tfrac{4}{3}\pi r^3, \text{ or } V = \tfrac{1}{6}\pi d^3.$$

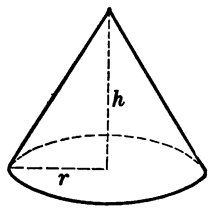

FIG. 16

18. Volume V of a spherical segment (a slice of a sphere between two parallel cutting planes), where h is the altitude and a and b the radii of the two bases. (Fig. 17.)

$$V = \frac{\pi h}{2}\left[(a^2 + b^2) + \frac{h^2}{3}\right]$$

19. Surface S of a zone (the portion of the surface of a sphere lying between two parallel cutting planes), where h is the distance between the cutting planes and r the radius of the sphere.

$$S = 2\pi r h.$$

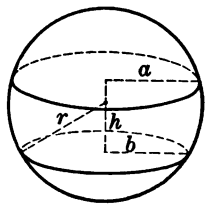

FIG. 17

EXERCISES AND PROBLEMS

1. The length of a rectangle is a. The length is three times the height. Express the area as a function of a.

2. The altitude of a triangle is 2 inches less than the base. Express the area in terms of the base.

3. The sum of the two bases of a trapezoid equals three times the altitude h. Express the area as a function of the altitude.

4. Express the area of a circle as a function of the circumference c.

5. The sides of a rectangular solid are a, $a+2$, $a+3$. Express the volume as a function of a.

6. The lengths of the sides of a rectangular solid are consecutive integers. The shorter side is of length l. Let V be the volume, then $V = f(l)$. Find $f(l)$.

7. The base of a pyramid is a square whose side equals the altitude h. Express the volume of the pyramid as a function of the altitude.

8. Find the volume of a pyramid of altitude 10 and square base of side 4.

9. Find the volume of a cone of altitude 7 and radius of base 4.

10. The altitude of a cone is one greater than three times the radius r of the base. Express the volume as a function of the radius.

11. Express the surface of a sphere in terms of the circumference C of a great circle (the largest circle that can be drawn on the sphere).

12. Let V be the volume of a sphere, and c the circumference of a great circle. Then $V = f(c)$. Find $f(c)$.

13. Find the volume of a spherical shell. The thickness of the shell is one inch and outside diameter 10 inches.

14. Express the following sentence in functional symbols. The number N of eggs bought for a given amount of money depends upon the price c per dozen.

15. Express in functional symbols: If the age x of the applicant for life insurance is known, the premium P is fixed.

16. If d is the distance passed over in t hours by an aeroplane going 70 miles an hour, then $d = f(t)$. Find $f(t)$.

46. System of coördinates. Let $X'X$ and $Y'Y$ (Fig. 18) be two straight lines meeting at right angles. Let them be considered as two number scales with the point of intersection as the zero point of each. Let P be any point in the plane. From it drop perpendiculars to the two lines. Let x represent the perpendicular to $Y'Y$ and y the perpendicular to $X'X$. If P lies to the left of $Y'Y$, x is to be considered negative. If P lies below $X'X$, then y is negative. It is clear that no matter where P is in the plane, there corresponds to it one and only one pair of perpendiculars, x and y.

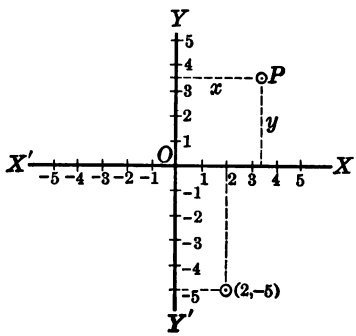

FIG. 18

The lines of reference $X'X$ and $Y'Y$ are called the **coördinate axes,** and their intersection is called the **origin.** The first line is called the X-axis, and the second the Y-axis. The perpendicular to the X-axis from a given point in the plane is called the **ordinate** or y value of the point. The perpendicular to the Y-axis is called the **abscissa** or x value of the point.

If we have two numbers given we can find one and only one point P which has the first number for its abscissa and the second for its ordinate. If, for example, the numbers are 2 and -5, we measure from the origin, in the positive direction a distance 2 on the X-axis and at this point we erect a perpendicular and measure downwards a distance 5. We have then located a point whose x is 2 and whose y is -5. This point may be represented by the symbol $(2, -5)$. The symbol (a, b)

denotes a point whose abscissa is a and whose ordinate is b. The symbol $P(a, b)$ is sometimes used and is read, "the point whose coördinates are a and b."

When a point is located in the manner described above, it is said to be **plotted.** In plotting points and obtaining the geometrical pictures we are about to make, it will be convenient to use coördinate paper (Fig. 19). Then the side of a square may be taken as the unit of length to represent a number.

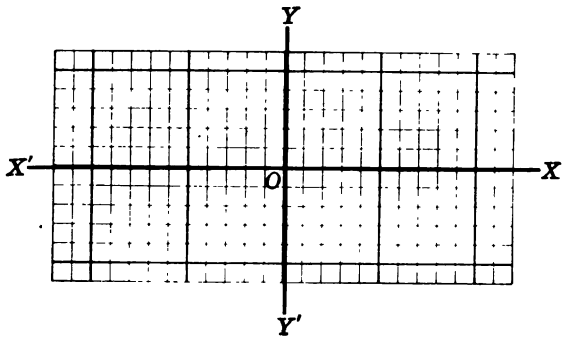

FIG. 19

To plot a point, count off from the origin along the X-axis the number of divisions required to represent the abscissa and from the point thus determined count off the number of divisions parallel to the Y-axis required to represent the ordinate.

EXERCISES AND PROBLEMS

1. Plot the points $(2, 4)$, $(-2, 4)$, $(2, -4)$, $(-2, -4)$.
2. Plot the points $(3, 7)$, $(7, 3)$, $(0, 0)$, $(0, 1)$, $(2, 0)$.
3. Plot the points $(-1, 1)$, $(-1, 0)$, $(\frac{1}{2}, \frac{1}{2})$, $(3, \frac{3}{4})$, $(-5, \frac{10}{3})$, $(-\frac{3}{2}, -\frac{7}{3})$, $(-\frac{5}{2}, 0)$.
4. Draw the triangle whose vertices are $(0, 0)$, $(0, 1)$, $(2, 0)$.
5. Draw the triangle whose vertices are $(1, 3)$, $(-1, -1)$, $(2, -2)$.

6. Draw the quadrilateral whose vertices are (0, 0), (2, 3), (5, −1), (3, −3).

7. A square of side 2 has one corner at the point (2, 1). If the sides of the square are parallel to the coördinate axes, what are the coördinates of the points that may be at the other corners of the square?

8. A line is bisected by the origin. One end of the line is the point (2, −5). What are the coördinates of the other end?

9. A circle of radius 2 with its center at the origin intersects the coördinate axes at what points? At what points does it intersect the straight line which passes through the points (2, 2), (−2, −2)?

10. Let the X-axis be considered as an east and west line, and the Y-axis as a north and south line. The following points on a river indicate its general course. Map the river from $x = -4$ to $x = +4$. $(-4, -2\frac{1}{4})$, $(-3\frac{1}{2}, -2\frac{1}{8})$, $(-3, -2)$, $(-2\frac{1}{2}, -1\frac{3}{4})$, $(-2, -1\frac{1}{2})$, $(-\frac{3}{2}, -1)$, $(-1, -\frac{1}{4})$, $(-\frac{1}{2}, \frac{3}{8})$, $(0, 1)$, $(\frac{1}{2}, \frac{3}{2})$, $(1, 1\frac{3}{8})$, $(\frac{3}{2}, \frac{3}{2})$, $(2, 1)$, $(2\frac{1}{2}, 0)$, $(3, -1)$, $(3\frac{1}{2}, -\frac{3}{2})$, $(4, -1\frac{3}{4})$.

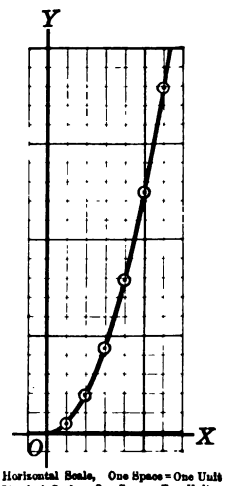

FIG. 20

47. Graph of a function. The change in a function can be represented to the eye on coördinate paper. For example, the change in the area of a square due to the change in the length of the side can be represented to the eye as follows: Let A be the area and a the length of the side. Construct a table showing the area for various values of a.

$a =$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
$A =$	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9

Draw coördinate axes on paper and plot the points $(\frac{1}{2}, \frac{1}{4})$, $(1, 1)$, $(\frac{3}{2}, 2\frac{1}{4})$, and so on (Fig. 20). Connect the points by a smooth curve.

The fact that the area increases more rapidly than the side is shown in the upward bending of the curve.

By this method, which is the same as that used to map a river (Exercise 10, Art. 46), any function may be represented on coördinate paper. This representation of a function is called the **graph** of the function. The graph of $f(x)$ contains all the points whose coördinates are $[x, f(x)]$ and no other points.

Example. Obtain the graph of $\frac{x}{2} - 1$ for values between -5 and $+5$.

Let $f(x) = \frac{x}{2} - 1$. The object is to present a picture which will exhibit the values of $f(x)$ which correspond to different values of x. Any value of x

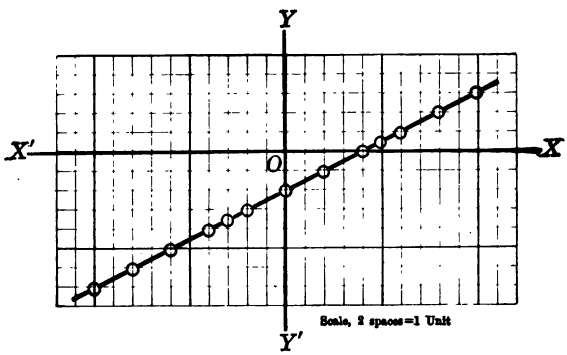

FIG. 21

with the corresponding value of $f(x)$ determines a point whose ordinate is $f(x)$. Assuming values of x and computing the corresponding values of $f(x)$, we obtain the following table.

$x =$	0	1	2	$2\frac{1}{2}$	3	4	5	-1	$-1\frac{1}{2}$	-2	-3	-4	-5
$f(x) =$	-1	$-\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\frac{3}{2}$	$-1\frac{3}{4}$	-2	$-\frac{5}{2}$	-3	$-3\frac{1}{2}$

The corresponding points $(0, -1)$, $(1, -\frac{1}{2})$, $\cdots$, are plotted in Fig. 21. It is seen from the figure that all the points lie on a straight line. This shows that the function $\frac{x}{2} - 1$ increases uniformly as x increases.

EXERCISES AND PROBLEMS

1. Draw a curve showing the change in the volume of a cube as the length of the edge x changes from 0 to 5.

2. One side of a rectangle is x, the other side is $x + 1$. Show by a curve the change in the area as x changes from 0 to 10.

3. Show by a graph the change of the circumference as the radius r changes.

4. Show by a graph the change in the area of a circle as the diameter d changes.

5. Exhibit graphically the change in the volume of a sphere as the diameter d changes.

Graph the following functions on coördinate paper from $x = -4$ to $x = 4$, plotting at least 17 points.

6. $x + 2$.
7. $2x + 1$.
8. $2x - 1$.
9. $\frac{x}{3}$.
10. x^2.
11. $x^2 + 1$.
12. $x^2 + x$.
13. $x^2 + x + 1$.
14. $1 + x - x^2$.
15. $1 - x - x^2$.

MISCELLANEOUS EXERCISES AND PROBLEMS

Evaluate the following functions:

1. $f(x) = \frac{3x + 2}{4}$, for $x = 1$, $x = 3$, $x = -1$, $x = 0$.
2. $f(s) = s^2 - 7$ for $s = -4$, $s = 1.2$, $s = 17$.
3. $f(m) = m^2 + 3m + 1$, for $m = -\frac{1}{2}$, $m = 13$, $m = -10$.
4. If $f(a) = a^4 + 2a^2 + a$, find $f(1), f(-1), f(0)$.
5. If $f(x) = \frac{x^2 + x + 1}{x^2 - x + 1}$, find $f(3), f(-3), f(a), f(10)$.

6. If $f(n) = \dfrac{1}{n^2+1}$, find $f(x^2), f(a+1)$.

7. If $f(x) = 2x - 1$, and $F(x) = 1 - 2x$, find $F(2) + f(2)$.

8. The dimensions of a rectangular solid are consecutive integers. The shortest edge is represented by x. Express the volume. Find the volume when $x = 7$; when $x = 11$.

9. Arrange the following numbers in order: $f(-3), f(-2), f(-1), f(0), f(\frac{1}{2}), f(\frac{3}{2}), f(2\frac{1}{2}), f(3)$, where $f(x) = x^2 + x + 1$.

10. Fill out the table where $f(a) = \dfrac{a^2+2a+3}{a^2+1}$.

$a =$	-3	-2	-1	0	1	2	3	4	5
$f(a) =$									

11. Fill out the table for $f(x) = 10x + 100x^2$.

$x =$	0	$.1$	$.2$	$.3$	$.4$	$.5$	$.6$	$.7$	$.8$	$.9$	1
$f(x) =$											

12. Fill out the table for $f(y) = \dfrac{1}{y}$.

$y =$	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$+\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3
$f(y) =$													

On coördinate paper graph the following functions from $x = -5$ to $x = 5$:

13. $\dfrac{x}{2}$.

14. $\dfrac{x}{2} - 1$.

15. $\dfrac{x-1}{2}$.

16. $x - x^2$.

17. $\dfrac{x+1}{x+6}$.

18. $(x+1)(x+2)$.

19. If $f(x) = x^4 - 2x^3 - x^2 + 2x$ show that $f(2) = f(1) = f(-1) = f(0)$.

20. For the function in Exercise 19, show that $f(3) = f(-2)$.

21. If $f(a) = a^2 + 1$, find $f(a+1)$.

22. If $f(x) = x^2 + x + 1$, find $f(2x-1)$.

23. If $f(x) = 2x + 1$, find $f(2x+1)$.

24. If $f(x) = x^2 - x - 1$, find $f\left(\frac{1}{x}\right)$.

25. If $F(x) = \frac{x+2}{x-2}$, find $F\left(\frac{1}{x}\right)$.

26. If $f(a) = \frac{a+2}{3a-1}$, find $f(x-2)$.

27. If F represents the length of an adult's foot in inches and s the size of the shoe, then

$$F = \frac{s}{3} + 8.$$

Represent by a diagram the relation between length of foot and size of shoe for sizes 1 to 12.

28. For the 14 days preceding the opening of the European war the daily average price of 20 well-known stocks was as follows: 100.70, 100.63, 100.01, 104.49, 98.30, 98.77, 98.49, 97.95, 97.05, 97.16, 96.58, 93.14, 94.12, 89.41. Show this decline in price by a diagram.

HINT: To save work in handling these large numbers, 80 may be subtracted from each of the above numbers and the differences plotted.

29. The weight of one mile of fine iron wire depends on the diameter of the wire according to the table:

Diameter in inches.	.007	.008	.009	.010	.011	.012	.013	.014	.015	.016	.017	.018	.019	.020
Weight in pounds.	.7	.9	1.1	1.4	1.7	2.0	2.4	2.7	3.1	3.6	4.0	4.5	5.0	5.6

Exhibit graphically.

30. The weight of water changes slightly with the temperature according to the table:

Temperature Fahrenheit.	40	50	60	70	80	90	100	110	120
Weight of cubic foot water in lbs.	62.42	62.41	62.37	62.31	62.28	62.13	62.02	61.89	61.74
Temperature Fahrenheit.	130	140	150	160	170	180	190	200	210
Weight of cubic foot water in lbs.	61.56	61.37	61.18	60.98	60.77	60.55	60.32	60.07	59.82

Exhibit graphically.

HINT: The number 60 may be subtracted from each of the above weights and the resulting graph will be unchanged in shape.

31. The weight which a manila rope will safely bear depends upon the diameter of the rope according to the table:

Diameter of rope in inches.	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3
Maximum safe load in tons.	$\frac{1}{4}$	1	2	$3\frac{1}{2}$	$5\frac{1}{4}$	$7\frac{1}{2}$	$10\frac{1}{4}$	$13\frac{1}{2}$	$17\frac{1}{4}$	21	25	$29\frac{1}{2}$

Represent graphically.

HINT: Let 2 inches be the horizontal unit and one quarter of an inch the vertical unit.

CHAPTER VI

SYSTEMS OF LINEAR EQUATIONS

48. Graphs of linear equations. The graphs of some functions are found in Chapter V.

REVIEW QUESTIONS

1. What is meant by (1) coördinate axes, (2) coördinates of a point?

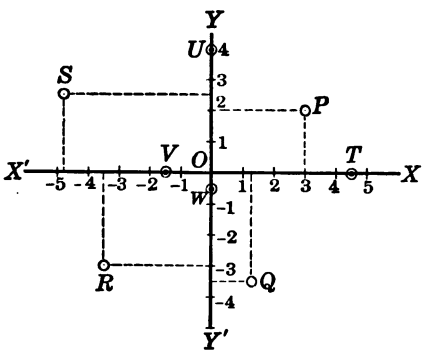

FIG. 22

2. Locate points represented by the symbols (2, 5), (−2, 5), (2, −5), (−2, −5).

3. What are the approximate coördinates of P, Q, R, S, T, U, V, and W in Fig. 22?

4. What is meant by the origin? What are its coördinates?

An equation of the form

$$ax + by + c = 0 \quad (1)$$

is called a **linear** equation. When $b \neq 0$, this equation may be put into the form

$$y = -\frac{ax}{b} - \frac{c}{b}. \quad (2)$$

Since in (2), for any given values of a and b, we may assign x and compute corresponding values of y, the equation gives any number of pairs of values to plot as coördinates of points. In other words, y is given as a function of x in (2), and the graph of this function is called the graph or **locus** of equation (1).

When a or b is zero, the graph is a line parallel to the X-axis or to the Y-axis. Thus, the equation

$$x + 2 = 0$$

gives a line parallel to the Y-axis, and 2 units to the left of that axis.

The linear equation is so called because its locus is a straight line.

EXERCISES AND PROBLEMS

Draw the loci of the following equations:

1. $x - y = 1$.

SOLUTION: Let $x = 0$, and we have $y = -1$. Let $y = 0$, and we have $x = 1$. The points $(0, -1)$, $(1, 0)$ being located, we draw a straight line through them. If we are given that the graph is a straight line, why is it sufficient to plot only two points? Assign further values, 1, 2, 3, 4, 5 to x and verify that the points determined fall on the locus plotted.

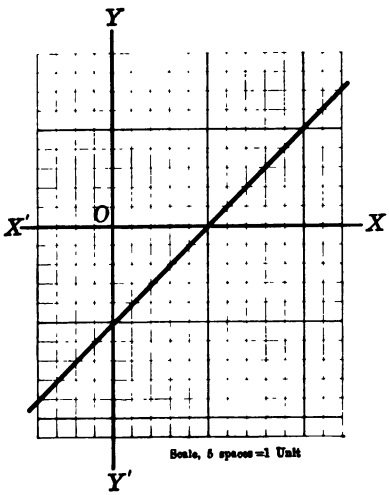

FIG. 23

2. $y - 2x = 4$.

3. $x - 2y = 2$.

4. $3x - 2y = 6$.

5. $3x + 2y = 6$.

6. $7x + 3y = 0$.

7. $7x + 3y = 4$.

8. $y - 2 = 0$.

HINT: Show the position of all points for which $y = 2$.

9. $x - 8 = 0$.

HINT: Show the position of all points for which $x = 8$.

10. The relation between temperature in degrees Fahrenheit F and degrees Centigrade C is given by

$$F = \frac{9}{5} C + 32.$$

Construct the locus of this equation.

11. The speed v in feet per second of a falling body at time t is given by

$$v = 32.2t + 10,$$

where 10 denotes the speed at the instant from which time is measured. Construct the locus that shows the relation.

12. Where does the locus $3x - y = 6$ cut the X-axis? The Y-axis? Answer the same question for the loci $2x - 4y = 9$, and $x + 3y = 0$.

13. Plot the loci of $x - y = 5$ and of $2x + y = 4$. Do the two lines have any point in common? What are the coördinates of this point? Do these coördinates satisfy both equations?

14. Plot $x - y = 5$ and $x - y = 10$. Do the two lines have a point in common?

15. Plot $x - y = 5$ and $2x - 2y = 10$. Do the graphs have points in common?

49. Graphical solution of a system of linear equations. As stated in Art. 48, the locus of any linear equation in x and y is a straight line. Any such equation is satisfied by an indefinitely large number of pairs of values of x and y; that is, by the coördinates of all points on the locus. In the graphical solution of the system of two equations, we seek the coördinates of points common to the loci of the two equations.

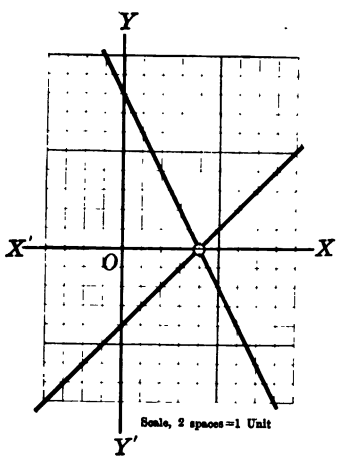

FIG. 24

As the loci are two straight lines, three cases arise:

(1) In general, two lines intersect in one and only one point. This situation is illustrated (Fig. 24) by the graphical solution of

$$x - y = 2,$$
$$y + 2x = 4.$$

The solution is $x = 2$, $y = 0$. Hence, in general, a system of two linear equations has one and only one solution, but two exceptions are now to be noted.

(2) Two lines may be parallel, and thus have no common point. This situation is illustrated (Fig. 25) by the graphs of the equations of the system

$$x - y = 2,$$
$$2x - 2y = 9.$$

When the loci are two parallel lines, there is no pair of numbers that satisfies both equations, and the equations are said to be **incompatible** or **inconsistent.**

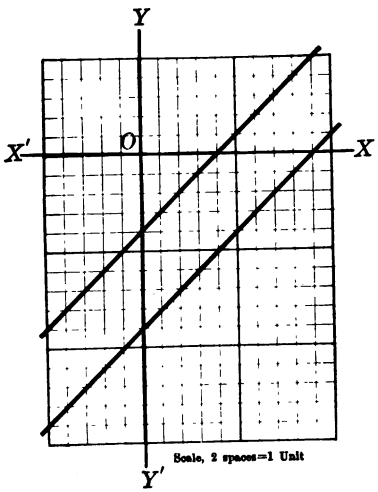

FIG. 25

(3) Two lines may be coincident, and thus have an indefinitely large number of points in common. Thus, if we plot $x - y = 2$, and $3x - 3y = 6$, we shall find that the graphs coincide. There are an indefinitely large number of pairs of values that satisfy this system. The two equations of the system are in this case **equivalent** or **dependent.**

EXERCISES

Find the solutions (if they exist) of the following equations by plotting loci:

1. $3x + y = 19$, $2x - y = 1$.
2. $2x + 3y = 4$, $x + 1.5y = 3$.
3. $2x + y = 5$, $6x - 5y = 23$.
4. $3x + 4y = 24$, $5x - 3y = 11$.
5. $x - 2 = 0$, $y - x = 4$.
6. $2x + 3y = 10$, $5x + 3y = 7$.

7. $7x + 8y = 15,$
$x - y = 0.$

8. $5x + 4y = 6,$
$5x + 4y = 13.$

9. Plot the locus of $y + 2x = 5$, then multiply both members by 3, and plot the locus of the resulting equation. Compare the graphs.

10. Plot the loci of $y + 2x = 5$, and of $3x - 2y = 4$. Then form an equation by adding the members of these two equations and plot the locus of the resulting equation. What are the coördinates of the point common to the three loci?

50. Elimination. The process of deriving from a system of n equations, a new system of $n - 1$ equations with one fewer unknowns, is called **elimination.** Thus, when the given system consists of two equations in two unknowns, the new system consists of a single equation in one unknown.

51. Elimination by addition or subtraction. Elimination by **addition or subtraction** consists first in multiplying the members of the two equations by such numbers as will make numerically equal the coefficients of the unknown to be eliminated. Then, by addition or subtraction of the members, this unknown is eliminated.

Example. Solve
$$6x - 5y = 14, \quad (1)$$
$$7x + 2y = 32. \quad (2)$$

SOLUTION: Eliminate y as follows:

(1) · 2 * gives $12x - 10y = 28.$ (3)

(2) · 5 gives $35x + 10y = 160.$ (4)

(3) + (4) gives $47x = 188,$ (5)

$x = 4.$ (6)

Substituting 4 for x in (1) gives

$24 - 5y = 14,$ (7)

$-5y = -10,$

$y = 2.$ (8)

CHECK: Substituting 4 for x and 2 for y in (1) and (2) gives

$24 - 10 = 14,$ (9)

$28 + 4 = 32.$ (10)

* (1) · 2 means " the members of equation (1) multiplied by 2."

52. Elimination by substitution. Elimination by **substitution** consists in expressing one unknown in terms of the other in one equation, and substituting this result in the other equation, thus obtaining one equation in one unknown.

Example. Solve $7x - 3y = 26,$ (1)

$2x + 11y = 43.$ (2)

SOLUTION: From (2), $2x = 43 - 11y,$ (3)

and $x = \frac{43 - 11y}{2}.$ (4)

Substituting $\frac{43 - 11y}{2}$ for x in (1) gives

$$7 \cdot \frac{43 - 11y}{2} - 3y = 26. \quad (5)$$

(5) · 2 gives $7(43 - 11y) - 6y = 52.$ (6)

Then $-83y = -249,$

$y = 3.$ (7)

Substituting 3 for y in (4) gives $x = 5.$ (8)

CHECK: Substituting 5 for x and 3 for y in (1) and (2) gives

$35 - 9 = 26,$ (9)

$10 + 33 = 43.$ (10)

EXERCISES

Solve the following systems of equations, and check the results:

1. $6x + 5y = 22,$
   $4x - y = 6.$
2. $x + 2y = 14,$
   $2x + 3y = 23.$
3. $2x + 5y = 4,$
   $4x - 10y = 48.$
4. $3x + 2y + 5 = 0,$
   $6x + 10y + 16 = 0.$
5. $2(x + y) = 16,$
   $3(x - y) = 6.$
6. $x + \frac{y}{5} = 17,$
   $x - y = 5.$
7. $\frac{x}{3} + 3y = 15,$
   $\frac{y}{4} + 4x = 37.$
8. $\frac{x}{3} + 5y + 1 = 0,$
   $\frac{x}{2} + y + 1 = 0.$

9. $0.8x + 0.1y = 0.19,$
$0.6x + 0.9y = 0.39.$

10. $\frac{x+5}{2} + 5y = 9,$
$\frac{y+3}{2} + x = 5.$

11. $ax + by = 2ab,$
$bx + ay = a^2 + b^2.$

12. $x + y = m + n,$
$mx - ny = m^2 - n^2.$

53. Systems of linear equations solved by determinants. Let

$$a_1x + b_1y = c_1, \tag{1}$$
$$a_2x + b_2y = c_2, \tag{2}$$

be a system of linear equations in two unknowns. We set the problem of finding values of x and y that will satisfy both equations. Multiply the members of (1) by b_2 and those of (2) by $-b_1$. Adding the members of the resulting equations, we obtain

$$(a_1b_2 - a_2b_1)x = b_2c_1 - b_1c_2,$$

or
$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

In a similar manner, by multiplying (1) by $-a_2$ and (2) by a_1, and adding, we obtain

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}, \text{ provided } a_1b_2 - a_2b_1 \neq 0.$$

We note that the denominators of the above fractions are alike. This denominator may be conveniently denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

which is called a **determinant.** Since it has two rows and two columns, it is said to be of the second order. The letters a_1, b_1, a_2, b_2, are called the **elements** of the determinant, and a_1, b_2 are said to constitute the **principal diagonal.** A determinant of the second order then represents the number which is ob-

tained by subtracting from the product of the terms in the principal diagonal, the product of the other two terms. Thus,

$$\begin{vmatrix} x & y \\ z & w \end{vmatrix} = xw - yz, \qquad \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

Although we have solved systems of linear equations by certain methods of elimination (Arts. 51, 52) it is often better to employ determinants. The use of determinants introduces us to an important instrument that is essential in the study of more advanced algebra.

Using the determinant notation, we may now write the solutions of our equations in the form

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

We note that the numerator of the solution for x is obtained from the denominator by substituting in place of a_1, a_2, which are the coefficients of x in the equations to be solved, the known terms c_1, c_2. In a similar manner, in the numerator of the solution for y we replace b_1, b_2 by c_1, c_2, respectively.

EXERCISES

1. Solve $x + y = 3$,
$2x + 3y = 1$.

SOLUTION:

$$x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{9 - 1}{3 - 2} = 8, \qquad y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{1 - 6}{3 - 2} = -5.$$

Solve the following pairs of equations, using determinants:

2. $7x + 9y = 41$,
$x + 4y = 14$.

3. $\frac{x}{3} + \frac{y}{5} = 9$,
$x + y = 35$.

4. $3x - y + 1 = 0.$
$3y - 4x + 10 = 0.$

5. $5x - 2y = 26,$
$4x + 3y = 7.$

6. $\dfrac{11x - 5y}{22} = \dfrac{3x + y}{32},$
$8x - 5y = 1.$

HINT: First clear of fractions and simplify.

7. $\dfrac{4x - 3y - 7}{5} = \dfrac{3x}{10} - \dfrac{2y}{15} - \dfrac{5}{6},$

$\dfrac{y - 1}{3} + \dfrac{x}{2} + \dfrac{3y}{20} - 1 = \dfrac{y - x}{15} + \dfrac{x}{6} + \dfrac{1}{10}.$

8. $ax - by = 0,$
$x - y = c.$

9. $4(x - 1) = 3 + 3x - \dfrac{y}{2},$
$2x - 3y + 2 = 0.$

10. $a(x + y) + b(x - y) = a^2 + b^2,$
$a^2(x + y) + b^2(y - x) = a^3 - b^3.$

11. $(m + n)x - (m - n)y = 21m,$
$(m + 1)x - (m - 1)y = 2mn.$

12. $x - 3y = 6,$
$4x - 5y = 24.$

13. $3x + 4y = 7 - 5y,$
$x - y = 6 - 2x.$

14. $0.3x + 0.2y = 4,$
$0.7x - 0.6y = 4.$

15. $mx + ny = 2mn,$
$nx + my = m^2 + n^2.$

54. Determinants of the third order. The square array of nine numbers with bars on the sides

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is a convenient abbreviation for the expression

$$a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1, \qquad (1)$$

and is called a determinant of the third order. As in the case of the determinant of the second order, the letters a_1, b_1, $\cdots$

are called the elements, and the letters a_1, b_2, c_3, form the principal diagonal. The expression (1) is called the **expansion** or **development** of the determinant. It is seen that each term of the expansion consists of the product of three elements, no two of which lie in the same row or in the same column. Any determinant of the third order may be easily expanded as follows. Rewrite the first and second columns to the right of the determinant. From the diagonals

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}$$

running down from left to right we find the first three terms of (1). From the diagonals running up from left to right we find the last three terms of (1), after prefixing negative signs.

EXERCISES

Obtain the expansions of the following determinants:

1. $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 1 & 3 & 5 \end{vmatrix}$

When the first and second columns are written to the right of the determinant, we have

$$\begin{vmatrix} 1 & 3 & 4 \\ 2 & 7 & 3 \\ 1 & 3 & 5 \end{vmatrix} \begin{matrix} 1 & 3 \\ 2 & 7 \\ 1 & 3 \end{matrix}.$$

The diagonals running down from left to right are

$$1, 7, 5;\ 3, 3, 1;\ 4, 2, 3.$$

The diagonals running up from left to right are

$$1, 7, 4;\ 3, 3, 1;\ 5, 2, 3.$$

The expansion then is

$$1\cdot7\cdot5 + 3\cdot3\cdot1 + 4\cdot2\cdot3 - 1\cdot7\cdot4 - 3\cdot3\cdot1 - 5\cdot2\cdot3 = 1.$$

2. $\begin{vmatrix} 1 & 2 & 8 \\ 3 & 3 & 12 \\ 3 & 1 & 1 \end{vmatrix}$ **5.** $\begin{vmatrix} 1 & x & y \\ 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & -4 \\ 8 & 0 & 5 \end{vmatrix}$ **6.** $\begin{vmatrix} a & b & 0 \\ 0 & x & y \\ u & 0 & 1 \end{vmatrix}$

4. $\begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$ **7.** $\begin{vmatrix} x & 1 & 3 \\ 2 & x & 2 \\ 6 & 3 & 1 \end{vmatrix}$

8. $\begin{vmatrix} x & 2 & y \\ 1 & 1 & 1 \\ x & 4 & y \end{vmatrix}$

55. Solution of three equations with three unknowns. Let the three equations be

$$a_1x + b_1y + c_1z = d_1, \tag{1}$$
$$a_2x + b_2y + c_2z = d_2, \tag{2}$$
$$a_3x + b_3y + c_3z = d_3. \tag{3}$$

Multiplying (1) and (2) by b_2 and $-b_1$ respectively and adding, we get

$$(a_1b_2 - a_2b_1)x + (c_1b_2 - b_1c_2)z = b_2d_1 - b_1d_2. \tag{4}$$

Eliminating y in a similar manner from (1) and (3), we find

$$(a_3b_1 - a_1b_3)x + (b_1c_3 - b_3c_1)z = b_1d_3 - b_3d_1. \tag{5}$$

We now have two equations in two unknowns x and z. Eliminating z from these two, we find

$$[(a_1b_2 - a_2b_1)(b_1c_3 - b_3c_1) - (a_3b_1 - a_1b_3)(b_2c_1 - b_1c_2)]x$$
$$= (d_1b_2 - d_2b_1)(c_3b_1 - b_3c_1) - (d_3b_1 - d_1b_3)(b_2c_1 - b_1c_2),$$

which after some simplification gives us

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_3b_2c_1 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_3b_2c_1 - a_2b_1c_3}.$$

The denominator is the development of the determinant in Art. 54, while the numerator is the same as the denominator with a replaced by d. Hence, we can write the solution for x in the form

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

provided the determinant in the denominator is not zero.

In a similar way, we can find the value of y and of z.

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \qquad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The denominators in the expressions for x, y, and z are the same, while the numerators are obtained from the denominators by replacing the coefficients of the unknown in question by the known terms. For example, in the numerator of y, the knowns d_1, d_2, d_3 replace b_1, b_2, b_3, respectively.

Historical note on determinants. It appears that the German philosopher and mathematician Leibnitz first used determinants in solving equations. In a letter to a friend, written in 1693, he introduced the notion of determinants in his efforts to simplify expressions arising in the elimination of certain unknowns from systems of equations. It seems, however, that Leibnitz made very little use of the method and that it did not become known to mathematicians; for, in 1750 Cramer, a professor at Geneva, rediscovered the method, and his work is accepted as forming the beginning of the development of the subject.

EXERCISES

Solve:

1. $x - y - z = -6,$
$2x + y + z = 0,$
$3x - 5y + 8z = 13.$

SOLUTION:

$$x = \frac{\begin{vmatrix} -6 & -1 & -1 \\ 0 & 1 & 1 \\ 13 & -5 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{-78}{39} = -2,$$

$$y = \frac{\begin{vmatrix} 1 & -6 & -1 \\ 2 & 0 & 1 \\ 3 & 13 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{39}{39} = 1, \qquad z = \frac{\begin{vmatrix} 1 & -1 & -6 \\ 2 & 1 & 0 \\ 3 & -5 & 13 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & -1 \\ 2 & 1 & 1 \\ 3 & -5 & 8 \end{vmatrix}} = \frac{117}{39} = 3.$$

2. $3x + 2y - z = 4,$
$5x - 3y + 2z = 5,$
$6x - 4y + 3z = 7.$

3. $x + y + z = 1,$
$3x + 2y + 7z = 1,$
$15x - 4y + 8z = 18.$

4. $2x + 3y + 5z = 2,$
$5x - y + 4z = 5,$
$7x - 2y + 6z = 5.$

5. $x + 2y + z = 0,$
$2x + y + 2z = 3,$
$4x - 6y + 3z = 14.$

6. $x + y = 1,$
$y + z = 2,$
$z + x = 4.$

7. $\frac{14}{x} + \frac{14}{y} = 23,$
$\frac{2}{y} + \frac{2}{z} = 5,$
$\frac{7}{x} + \frac{7}{z} = 8.$

8. $ax + by = a,$
$by + cz = b,$
$ax + cz = c.$

9. $ax + by - cz = 2ab,$
$by - ax + cz = 2bc,$
$ax - by + cz = 2ac.$

10. $x + y = 3a,$
$x + z = 4a,$
$y + z = 5a.$

11. $\frac{x}{2} + \frac{y}{3} = 9,$

$\frac{x}{3} + \frac{z}{2} = 8,$

$\frac{y}{2} + \frac{z}{3} = 13.$

12. $A - 5B - 6C = 2,$

$-3A - 4B + 10C = 7,$

$10A + 2B - 36C = -1.$

PROBLEMS

1. The sum of two numbers is 95, and their difference is 15. Find the numbers.

2. Thirty-four tons of slate are to be hauled in two-horse and four-horse loads. It is found that this will require 9 two-horse and 4 four-horse loads. Assuming a four-horse load is twice a two-horse load, how much is carried on each kind of load?

3. A workman is engaged for 30 days on the following terms. He is to receive \$3 for each day that he works, and is to be fined \$1.50 for each day he is idle. He received \$63 for the 30 days. How many days did he work?

4. Two books cost a cents. The one cost $\frac{a}{5}$ cents more than the other. Find the cost of each book.

5. A was m times as old as B a years ago, and will be n times as old as B in b years from now. Find the ages of each in terms of a, b, m, and n.

PROBLEMS PERTAINING TO DIGITS

6. Two numbers are written with the same two digits; the difference of the two numbers is 45 and the sum of the digits is 9. What are the numbers?

7. The sum of the two digits with which a number is written is 15. The digit in tens' place is one larger than that in units' place. Find the number.

8. There is a number of three digits whose sum is 14. The sum of the digits in tens' and hundreds' places is equal to that in units' place. If the digits were written in reverse order, the resulting number would be 495 greater than the given number. Find the number.

9. There is a number of three digits whose sum is 12. The digits in tens' and units' places are equal. That in hundreds' place equals twice that in units' place. Find the number.

PROBLEMS PERTAINING TO MIXTURES

10. What quantities of silver 72% pure and 84.8% pure must be mixed together to give 8 ounces of silver 80% pure?

11. What quantities of two liquids, one 95% alcohol and the other 15% alcohol, must be used to give a 10-gallon mixture of 45% alcohol?

12. The crown of Hiero of Syracuse was part gold and part silver. It weighed 20 pounds, and lost 1.25 pounds when weighed in water. How much gold and how much silver did it contain if 19.25 pounds of gold and 10.5 pounds of silver each lose a pound when weighed in water?

EXPLANATION OF WEIGHT IN WATER. A body like a piece of gold or silver when weighed in water loses an amount of weight equal to the weight of the water displaced. Thus, if x and y denote the weights of gold and silver,

$$\frac{x}{19.25} + \frac{y}{10.5} = 1.25.$$

13. One bar of metal is 20% pure silver and another is 12% pure silver. How many ounces of each bar must be used, if, when the parts taken are melted together, a bar weighing 40 ounces is obtained, of which 15% is pure silver?

PROBLEMS PERTAINING TO BUSINESS

14. A man has \$35,000 at interest. For one part he receives $3\frac{1}{2}\%$ and for the other 4%. His income from the money is \$1300 per year. How is the money divided?

15. What is the capital of a person whose income is \$1302, when he has $\frac{1}{3}$ of it invested at 8%, $\frac{1}{8}$ at 6%, and the remainder at 5%?

16. The garrison of a certain town consists of 240 men who receive \$3850 as monthly pay. The monthly pay of a cavalryman is \$20 and of an infantryman \$15. How many men of each class are there in the garrison?

17. A farm laborer engaged to work 90 days at \$1.50 per day and board. For days which he was idle, he was to pay \$1 for board. At the end of the time he received \$97.50 net. How many days did he work?

18. A man distributed a dollars among n persons of two classes, the one class receiving b dollars each, and the other c dollars each. Find in terms of a, b, and c the number that received b dollars and the number that received c dollars.

PROBLEMS PERTAINING TO AVERAGES

19. Four numbers have the property, that when successively the arithmetical average of three of them is added to the fourth, the numbers 29, 23, 21, 17 result. What are the numbers?

20. A student has three grades which give an average 85. The sum of two extreme grades exceeds the intermediate grade by 75. The highest grade exceeds the intermediate by 3. Find the grades.

21. A high school student received the same grade in solid geometry and in second-year algebra, and 3% less in plane geometry than in first-year algebra. To find the average grade of the student in all his mathematics, the grades in second-year algebra and in solid geometry were each multiplied by $\frac{1}{2}$, and

added to the grades in first-year algebra and plane geometry. The sum thus obtained was divided by 3. The average thus computed equalled 87. If the grades in first and second year algebra were interchanged, the average would be decreased by exactly 1 %. Find the grades.

PROBLEMS PERTAINING TO MENSURATION

22. The three angles of a triangle are together equal to 180°. The largest angle is 4 times as large as the smallest one, and equal to the sum of the two smaller angles. Find the three angles.

23. The perimeter of a rectangle is 330 inches. If the length of the longer side is doubled, the perimeter is 528. What are the dimensions of the rectangle?

24. The sums of the three pairs of adjacent sides of a triangle are 180, 200, and 220. Find the sides of the triangle.

25. The area of a trapezoid is equal to one-half the product of the sum of the two bases by the altitude. The altitude is 6, and the area is 54. The lower base is 2 longer than the upper base. Find the upper and the lower base.

PROBLEMS PERTAINING TO PHYSICS

26. In Wilson and Gray's determination of the temperature of the sun the Fahrenheit reading of the temperature is 5552 more than the Centigrade reading. What is the Centigrade reading. See Exercise 10, Art. 48.

27. It is required to find the amount of expansion of a brass rod for a rise in temperature of 1 degree Centigrade, also the length of the rod at temperature 0°. If c represents the expansion, and b_0 the length required, it is known that $b = ct + b_0$, where b is the length of the rod at temperature t. When $t = 20°$, the length of the rod is 1000.22 ; when $t = 60°$, the length is 1001.65.

28. If h represents the height in meters above sea level, and b represents the reading of a barometer in millimeters, it is

known that $b = k + hm$, where k and m are constants to be determined from observation. It is observed that at height 120 meters, the barometer reads 751, at height 769 meters it reads 695. Determine k and m. Use the resulting formula to calculate the barometer reading for a height of 1000 meters.

29. Two boys, Jack and Bill, wish to determine their weights by means of a teeter board. They find the board balances when Jack is 8 feet from the fulcrum and Bill is 6 feet from it. When Jack takes a 5-pound weight in his hand, he has to move $\frac{1}{2}$ foot nearer the fulcrum to maintain the balance. Find the weight of each of the boys. (See p. 22.)

30. A weight of 400 pounds is carried by two men by means of a pole, at a certain point of which the weight is hung. One man holds the pole at a distance $3\frac{1}{2}$ feet from the weight and the other at $4\frac{1}{2}$ feet. What amount of the weight does each man lift?

31. Two unknown weights balance when placed 8 and 12 inches from a fulcrum; if their positions are reversed, 25 pounds must be added on the side of too small leverage in order to balance. What are the weights?

56. Indeterminate equations. In this chapter, we have been engaged mainly in solving systems of equations where there are as many equations as unknowns. However, in Art. 49 attention is called to the fact that a single linear equation in two unknowns has an unlimited number of solutions. In general, when the number of equations in a system is fewer than the number of unknowns, we have what are called **indeterminate equations.**

In this book, we shall consider only the case of a single equation in two unknowns. Thus,

$$x + 2y = 4 \tag{1}$$

is satisfied by

$$\begin{array}{lll} x = 0, & x = 1, & x = 2, \\ y = 2, & y = \frac{3}{2}, & y = 1, \end{array}$$ and so on.

It is sometimes desirable to solve such an equation for integral values or for positive integral values of the unknowns.

EXERCISES AND PROBLEMS

1. Show that the equation $3x + 6y = 5$ has no integral solutions.

SOLUTION: Since $3x + 6y = 5,$ (1)

$3x = 5 - 6y,$ (2)

and $x = \frac{5}{3} - 2y.$ (3)

When integral values are assigned to y, we get no integral values for x.

2. A farmer spent \$36 buying two kinds of sheep, the one kind costing \$3 per head and the other kind \$5 per head. How many of each did he buy?

SOLUTION: Let x = number of sheep at \$3 and y the number at \$5.

Then $3x + 5y = 36,$

$3x = -5y + 36,$

$x = 12 - \frac{5}{3}y.$

To give integral values greater than 0, y can take values 3, and 6.

Hence, 7 head at \$3 and 3 head at \$5,

or 2 head at \$3 and 6 head at \$5.

3. How many whole watermelons at 35 cents and whole cantaloupes at 15 cents can be purchased for \$2.10?

4. Separate 37 into two parts one of which is divisible by 5 and the other by 11.

5. How many selections of coins can be made to pay a debt of 60 cents with quarters and dimes?

6. I desire to weigh a mass of 15 ounces by means of two-ounce and three-ounce weights placed in the same pan. How can this be done?

Historical note on indeterminate equations. It may be of interest to note that the early work in algebra centered on indeterminate equations rather than on determinate equations. The writings of the Greek alge-

braist Diophantus of Alexandria (about 350 A.D.) were devoted largely to the solution of indeterminate equations. He did not, however, devise general methods for the solution of indeterminate equations. Each problem had its own peculiar method, and it took great ingenuity to devise solutions for the many particular cases.

The credit for the invention of general methods of treating indeterminate equations belongs to the Hindus. The Hindus set the problems of indeterminate analysis in a different form from that of the Greeks. The object of the former was to find all possible integral solutions while Diophantus was content with a single rational answer.

CHAPTER VII

RATIO, PROPORTION, AND VARIATION

57. Ratio. The **ratio** of a number a to a number b is the quotient $\frac{a}{b}$ obtained by dividing a by b. The ratio a to b is also written $a : b$.

It is clear from the above definition that any ratio is a fraction and any fraction may be regarded as a ratio. Thus, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{c}{d}$ are ratios.

58. Ratios involved in measurement. It is good usage and often convenient to speak of the ratio of two quantities if they have a common unit of measure. Thus, the ratio of 6 feet to 2 feet is $\frac{6}{2}$.

To measure a quantity is to find its ratio to a given unit of measure. Thus, when we say a bar is 3 yards long, we mean that the ratio of the length of this bar to that of the standard yard is 3.

EXERCISES

Express the following ratios as fractions and simplify:

1. $6 : 8$.
2. $15 : 10$.
3. $2\frac{3}{8} : 8$.
4. $(x^2 - y^2) : (x - y)^2$.
5. $\left(1 - \frac{1}{a^2}\right) : \left(1 + \frac{1}{a}\right)$.
6. 10 feet : 6 feet.
7. $3\frac{1}{2} : 5\frac{1}{2}$.
8. $(x^2 - y^2) : (x + y)$.
9. $\frac{1}{x - 1} : \frac{1}{x^2 - 3x + 2}$.

10. Separate the number 168 into two parts in the ratio of 3 : 4.

11. Two numbers are in ratio 2 to 5, and if 1 is added to each number, they are in ratio 3 to 7. Find the numbers.

12. Separate the number 168 into three parts that are to each other as 1 : 2 : 4.

HINT: Let x, $2x$, and $4x$ be the parts.

59. Proportion. A **proportion** is a statement of the equality of two ratios. Thus,

$$\frac{a}{b} = \frac{c}{d}$$

is a proportion and is often written

$$a : b = c : d.$$

It is read "a is to b as c is to d."

The four numbers a, b, c, and d are said to be in proportion, a and d being called the **extremes** and b and c the **means** of the proportion.

EXERCISES

Find the value of x in the following proportions:

1. $\frac{x}{5} = \frac{7}{10}$.

2. $\frac{5}{x} = \frac{7}{10}$.

3. $x : 12 = 2 : 3$.

4. $\frac{3}{4} = \frac{x}{16}$.

5. $\frac{3}{5} = \frac{12}{x}$.

6. $4 : 7 = x : 14$.

7. If $ad = bc$, show that $\frac{a}{b} = \frac{c}{d}$.

8. If $ad = bc$, show that $\frac{b}{a} = \frac{d}{c}$.

9. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{a} = \frac{c+d}{c}$.

10. If $\frac{a}{b} = \frac{c}{x}$, then x is said to be a **fourth proportional** to a, b, and c. Find a fourth proportional to the following sets of numbers:

(a) 5, 10, -6.
(b) 4, 3, -12.
(c) 4, -6, -5.

11. If $\frac{a}{x} = \frac{x}{d}$, then x is said to be a **mean proportional** between a and d. Find the mean proportional between the following sets of numbers:

(a) 4 and 9.
(b) $+3$ and $+48$.
(c) -3 and -48.

12. If $\frac{a}{b} = \frac{b}{x}$, x is said to be a **third proportional** to a and b. Find a third proportional to the following pairs of numbers:

(a) 2, 3.
(b) 2, -3.
(c) 7, 11.
(d) -7, -11.

13. Find two numbers in the ratio of 3 to 4 whose sum is 28.

14. Find two numbers in the ratio 3 to -4 whose sum is -15.

15. What number must be added to each term of the fraction $\frac{1}{10}$ to give a fraction equal to $\frac{2}{5}$?

16. What number must be added to each term of $\frac{5}{9}$ to obtain $\frac{2}{3}$?

Write as proportions the following:

17. $6 \cdot 8 = 2 \cdot 24$.

18. $5 \cdot 8 = 4 \cdot 10$.

19. $3 \cdot 5 = x \cdot 4$.

20. $ab = 4 \cdot 3$.

21. $a^2 - b^2 = 3 \cdot 5$.

22. $x^2 - 3x + 2 = ab$.

23. $xy = 25$.

24. $x = \dfrac{ab}{c}$.

25. $ab = cx$.

26. $4x = 6 \cdot 7$.

27. $a = b$.

SOLUTION: $\dfrac{a}{1} = \dfrac{b}{1}$.

28. $4 = 2 \cdot 2$.

29. $x = y$.

Solve the following for x:

30. $2 : x = 5 : 8$.

31. $x : \frac{1}{2} = 28 : 2$.

32. $(x - 2) : 2 = 2 : \frac{1}{3}$.

33. $3 : x = x : 27$.

34. $4 : 7 = \dfrac{1}{x} : 3$.

35. $x : x - 1 = 3 : 2$.

36. $\frac{1}{2} : x = x : 10 - x$.

37. $(1 - x) : (1 + x) = (2 - x) : (2 + x)$.

60. Properties of proportions.

I. *In any proportion, the product of the means equals the product of the extremes.* For, in

$$\frac{a}{b} = \frac{c}{d},$$

when we multiply each member by bd, we obtain

$$ad = bc.$$

II. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{a}{c} = \dfrac{b}{d}$.

HINT: Multiply both members by $\dfrac{b}{c}$.

That is, in a proportion the means may be interchanged without destroying the equality.

In this case, the second proportion is said to be obtained from the first by **alternation.**

III. If $\dfrac{a}{b} = \dfrac{c}{d}$, show that $\dfrac{b}{a} = \dfrac{d}{c}$.

HINT: Divide the members of $1 = 1$ by the members of the equation.

In this case, the second proportion is said to be obtained from the first by **inversion.**

IV. If $\frac{a}{b} = \frac{c}{d}$, show that $\frac{a+b}{b} = \frac{c+d}{d}$.

HINT: Add 1 to each member and obtain $\frac{a}{b} + 1 = \frac{c}{d} + 1$.

In this case, the second proportion is said to be obtained from the first by **composition.**

V. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a-b}{b} = \frac{c-d}{d}$.

HINT: Subtract 1 from each member.

In this case, the second proportion is said to be obtained from the first by **division.**

VI. If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

HINT: Divide members of $\frac{a+b}{b} = \frac{c+d}{d}$ in IV by the members of $\frac{a-b}{b} = \frac{c-d}{d}$ in V.

In this case, the proportion $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ is said to be obtained from $\frac{a}{b} = \frac{c}{d}$ by **composition and division.**

In this connection, instead of the words "composition," "division," "composition and division," some authors use the words "addition," "subtraction," and "addition and subtraction" respectively.

EXERCISES

From the following proportions, find the proportions that arise by alternation, inversion, composition, division, and composition and division.

1. $2:4 = 3:6$.
2. $\frac{1}{2}:3 = \frac{7}{2}:21$.
3. $7:\frac{3}{2} = -21:-\frac{9}{2}$.
4. $x:y = 2:3$.
5. $x:x+1 = 3:7$.
6. $(x+y):(x-y) = 3:5$.

If $a : b = c : d$, prove

7. $\frac{a}{2b} = \frac{c}{2d}$.

9. $\frac{a}{3b} = \frac{c}{3d}$.

8. $\frac{a+2b}{a-2b} = \frac{c+2d}{c-2d}$.

10. $\frac{a+3b}{a-3b} = \frac{c+3d}{c-3d}$.

11. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

HINT: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$, then $a = bk$, $c = dk$, $e = fk$.

12. If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4}$, show that

$$\frac{a_1 + a_2 + a_3 + a_4}{b_1 + b_2 + b_3 + b_4} = \frac{a_1}{b_1}.$$

PROBLEMS INVOLVING SIMILAR FIGURES

13. The sides of a triangle are 3, 4, and 5. In a similar triangle the shortest side is 5. What are the other sides? (Fig. 26.)

Similar figures are figures of the same shape. (Figs. 26, 27.) In two similar figures any two of the sides of one are proportional to the two corresponding sides of the second. The areas of similar figures have the same ratio as the squares of corresponding sides.

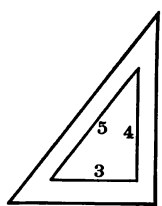

FIG. 26

14. The sides of a triangle are 9, 10, and 11. If the shortest side is lengthened one inch, what are the increases in the length of the other two sides in order to make the new triangle similar to the old?

15. In Fig. 27 the sides of the larger figure are 1, 2, 3, 4, 5. The longest side of the smaller figure is 3. What are the lengths of the other sides?

16. The area of a triangle whose base is 12 inches is 60 square

inches. If the area of a second triangle similar to the first is 135 square inches, what is the base of the second triangle?

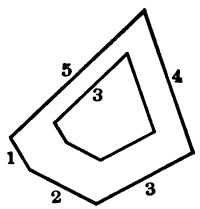

FIG. 27

17. A post 4 feet high, 20 feet from a street light, casts a shadow 7 feet long. What is the height of the light on the lamp post?

18. If in a map the distance between two points 450 miles apart is 4 inches, what is the distance between two cities which are $5\frac{1}{2}$ inches apart on the map?

19. The perimeter of a triangle is 78 inches, and the longest side is 9 times the shortest, while the third side is a mean proportional between the other two. Find the sides of the triangle.

PROBLEMS INVOLVING SIMPLE MACHINES

20. The arms of a lever are 6 and $4\frac{1}{2}$ feet. What weight on the end of the short arm will just balance a weight of 100 pounds on the end of the long arm?

For principle of the lever, see p. 23. The formula there given may be written as a proportion

$$W : F = f : w.$$

21. A pump handle is 4 feet long and works on an axis 4 inches from the pump rod end. What force is required at the end of the handle to overcome a pump rod resistance of 20 pounds?

22. Where must the support of a seesaw board 14 feet long be placed, in order that two children weighing 60 and 75 pounds may balance?

23. A crowbar 6 feet long is used to lift a stone weighing 800 pounds. The fulcrum of the crowbar is 6 inches from the end of the bar. What force must be applied at the end of the bar to lift the stone?

24. The length of the handles of a wheelbarrow is $4\frac{1}{2}$ feet. The center of a load of 200 pounds is 20 inches from the axle of the wheel. What force is required to lift the load?

25. In a wheel and axle, the radius of the wheel is 10 inches and the radius of the axle is 3 inches. What force applied to the rim of the wheel will wind up a weight of 140 pounds?

The wheel and axle (Fig. 28) consists of a wheel and cylinder fastened together and turning about the same axis. The force F applied at the circumference of the wheel to lift the weight W by a rope wound around the cylinder depends upon the radii r of the cylinder and r' of the wheel according to the formula,

$$F : W = r : r'.$$

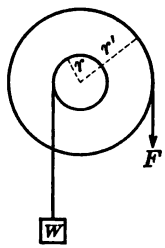

FIG. 28

26. The crank to a well windlass is 20 inches long and the cylinder upon which the rope is wound is 6 inches in diameter. What force is necessary to lift a bucket of water weighing 60 pounds?

27. After the cylinder in Problem 26 has been closely wound with rope and a second layer of rope has begun to be wound on the cylinder, how much is the force increased to lift 60 pounds if the rope is one inch in diameter?

28. A brakeman pulls with a force of 160 pounds on a brake wheel 18 inches in diameter to wind the brake chain about the axle of the wheel. If the axle is 4 inches in diameter what is the pull on the brake chain?

29. A safe weighing 900 pounds is pushed up an inclined plane by means of rollers. What force is necessary to keep it from slipping back if the inclined plane is 10 feet long and raised 2 feet at one end.

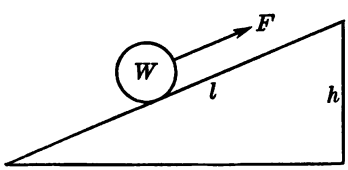

FIG. 29

The force F necessary to hold a weight W on an inclined plane (Fig. 29) is given by the proportion

$$F : W = h : l,$$

where l is the length of the plane and h the height through which one end is raised. No account is here taken of friction.

30. Find the force necessary to prevent a barrel weighing 350 pounds from rolling down a plane 21 feet long, one end of which is raised 4 feet.

31. If a boy can exert a maximum force of 100 pounds, what is the maximum height of the end of the inclined plane in Problem 30, so that the boy can handle the barrel?

61. Variation. In Chapter V we have seen that if y is a function of x, written

$$y = f(x),$$

then in general y changes when x changes. We may say that y varies when x varies, but the word "varies" has come to have a more restricted meaning when used in this connection. Each of the statements

"y varies as x,"
"y varies directly as x,"
"y is proportional to x,"
"y is directly proportional to x,"

means that y equals the product of x by a constant. That is,

$$y = kx.$$

The constant k is called the **constant of variation.**

The expression "y varies as x" is sometimes written

$$y \propto x.$$

The area of a circle varies as the square of its radius. That is,

$$A = kr^2,$$

if A represents the area and r the radius. With our restricted meaning of the word "varies," it is not correct to say that the area of a circle varies as the radius, for, in the equality $A = k \cdot r$, k is not a constant for different values of r.

If a train moves with a uniform speed, the distance s traversed varies as the time t. That is,

$$s = kt.$$

62. Inverse variation. Each of the statements

"y varies inversely as x,"
"y is inversely proportional to x,"

means that y is equal to the product of the reciprocal of x and a constant. That is,

$$y = \frac{k}{x}.$$

Thus, the volume of air in the cylinder of a bicycle pump varies inversely as the pressure on the piston. That is,

$$V = \frac{k}{p}$$

if V represents volume and p pressure.

63. Joint variation. The statement "z varies jointly as x and y" means that z equals the product of x, y and a constant. That is,

$$z = kxy.$$

The distance which a train, moving with a uniform speed, travels varies jointly as the speed and the time, or

$$d = kvt,$$

where d is the distance covered, v the speed, and t the time. In this case $k = 1$, if v and d are measured with the same unit of length.

64. Combined variation. The statement "z varies directly as x and inversely as y" means that z varies jointly as x and the reciprocal of y. That is,

$$z = \frac{kx}{y}.$$

If T varies directly as x, directly as the square of y, inversely as w and inversely as the cube of v, we have

$$T = k\frac{x\,y^2}{w\,v^3}.$$

The attraction F of any two masses m_1 and m_2 for each other varies as the product of the masses and inversely as the square of the distance r between the two bodies. That is,

$$F = \frac{km_1m_2}{r^2}.$$

EXERCISES AND PROBLEMS

Reduce each of the following statements, Exercises 1 to 7, to an equation:

1. y varies as x, and $y = 34$ when $x = 2$.

SOLUTION: $y = kx$.

Since $y = 34$ when $x = 2$, we have

$$34 = 2k, \text{ or } k = 17.$$

Hence, $y = 17x$.

2. T is directly proportional to t, and $t = 3$ when $T = 27$.

3. n varies inversely as m, and $n = 39$ when $m = 3$.

4. S varies as t^2, and $S = 144$ when $t = 3$.

5. A varies jointly as B and C. When $B = \frac{1}{2}$ and $C = 4$, it is found that $A = 14$.

6. P varies directly as q and inversely as r^2. When $q = 1$ and $r = \frac{1}{3}$, it is found that $P = 27$.

7. The surface S of a sphere varies directly as the square of the radius r. The surface of a sphere of radius 1 is 4π.

8. If x varies as y, and if $y = 1$ when $x = 3$, find x when $y = 5$.

SOLUTION: $x = ky$.

Since $y = 1$ when $x = 3$, we have

$$3 = k \cdot 1, \; k = 3.$$

or $x = 3y$.

When $y = 5$, we find $x = 15$.

9. If x varies as y and if $x = 3$, when $y = 6$, find y when $x = 7$.

10. If a varies inversely as b, and if $a = 3$ when $b = 1$, find a when $b = 8$.

11. If z varies jointly as x and y, and if $z = 120$ when $x = 2$ and $y = 3$, find z when $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

12. If z varies directly as x and inversely as y, and if $z = 120$ when $x = 2$ and $y = 3$, find z when $x = \frac{1}{2}$ and $y = \frac{1}{3}$.

13. If x varies directly as the square of y and if $x = 4$ when $y = 1$, find x when $y = 4$.

14. The safe load of a horizontal beam 10 feet long supported at both ends varies jointly as the breadth and square of the depth. If a 2 by 6 white pine joist 10 feet long safely holds up 800 pounds, what is the safe load of a 2 by 8 joist of same length?

15. Write in the form of an equation the law: The safe load w of a horizontal beam supported at both ends varies jointly as the breadth b and the square of the depth d and inversely as the length l between supports.

16. A beam 15 feet long, 3 inches wide, and 6 inches deep when supported at both ends can bear safely a maximum load of 1800 pounds. What is the maximum load for a beam of the same material 10 feet long, 2 inches wide and 4 inches deep?

17. What is the safe load for the second beam mentioned in Problem 16 if it is turned so that the width is 4 inches and depth 2 inches?

18. Write in the form of an equation the law: The crushing load of a solid square oak pillar varies directly as the fourth power of its thickness t and inversely as the square of its length l.

19. If a four-inch oak pillar 8 feet high is crushed by a weight of 100 tons, what weight will crush a pillar half as high and 6 inches thick? (See Problem 18.)

20. What weight will crush a four-inch oak pillar 4 feet high?

21. The deflection d of a rectangular beam of a fixed length varies inversely as the product of the breadth b and the cube of the depth d. Write this statement in the form of an equation.

22. In the formula

$$s = \frac{kbd}{l},$$

s denotes the strength of a rectangular beam, b, d, and l, the breadth, depth and length respectively, of the beam, and k is a constant. Translate the formula into English using the terms of variation.

PROBLEMS INVOLVING MOTION

23. The number of feet a body falls varies directly as the square of the number of seconds occupied in falling. If the body falls 16.1 feet the first second, how many feet will it fall in 6 seconds?

24. How far will a body fall during the sixth second?

25. The velocity of a falling body at any time varies directly as the number of seconds occupied in falling. What is the velocity at the end of 6 seconds if the velocity at the end of the first second is 32.2 feet per second?

26. An object dropped from a balloon strikes the ground in 7 seconds. At what velocity does the object strike the ground and what is the height of the balloon when the object is dropped?

27. A wrench is dropped from an automobile at a height of 3 feet while the automobile is traveling at the rate of 70 miles an hour. How far does the automobile move while the wrench is falling?

28. The time for one vibration of a pendulum at a given place varies as the square root of the length of the pendulum. In Chicago a pendulum 4 feet long requires 1.1 seconds for a vibration. What is the time of vibration of a pendulum 1 foot long?

29. What is the length of a pendulum which vibrates every second at Chicago?

30. A weight is suspended by a wire 94 feet long. What is the time of one vibration at Chicago?

31. A pendulum supposed to vibrate every second registers 90,000 vibrations in 24 hours. How much must the pendulum be lengthened?

PROBLEMS INVOLVING PRESSURE

32. The volume of a gas enclosed in a vessel varies inversely as the pressure upon it. Twenty-four cubic inches of air under a pressure of 100 pounds will have what volume when the pressure is decreased to 50 pounds?

33. If a toy balloon contains 150 cubic inches of gas when under a pressure of 15 pounds per square inch, to what size will it shrink if subjected to a pressure of 45 pounds per square inch? (See Problem 32.)

34. The pressure of wind on a sail varies jointly as the area of the sail and the square of the wind's velocity. When the velocity is 15 miles per hour, the pressure on a square foot is 1 pound. What is the velocity of the wind when the pressure is 10 pounds per square foot?

35. The pressure of gas in a tank varies jointly as its density and its absolute temperature. When the density is 1 and the temperature 300°, if the pressure is 15 pounds per square inch, what is the pressure when the density is 2 and the temperature 290°?

CHAPTER VIII

EXTENSION OF THE NUMBER CONCEPT

65. Integers and fractions. The first numbers studied in school are the numbers called positive integers. With these numbers the pupil is able to add and to multiply two numbers together and to subtract a smaller from a larger. He understands the answer, for it is always a positive integer. But division cannot always be performed if we have only positive integers; for example, the division of 4 by 3 is set aside as impossible.

A student knowing only the positive integers might learn to solve equations of some kinds. For example, the equations $4 - x = 0$, $3x - 6 = 0$, would give no difficulty, but the solution of the equation $2x - 5 = 0$ would be declared impossible, for the student knows no numbers which will satisfy the equation. He might call the result imaginary.

With the set of positive integers our elementary mathematics goes little further than counting; even the simple operation of measuring cannot be performed. For this reason early in his school life the number system of the pupil is increased by adding to it the numbers which we call fractions. These new numbers, while having no meaning in counting, have very definite practical applications.

EXERCISES

Assuming that we know only positive integers and zero, show which of the following exercises are possible of solution.

Solve:

1. $x - 3 = 0$.
2. $3x - 15 = 0$.
3. $3x + 9 = 0$.
4. $2x + 7 = 0$.

5. $5x + 4x = 0$.

6. $3x - 5 = 0$.

7. $x^2 - 3x + 2 = 0$.

8. $x^2 - 1 = 0$.

9. $x^2 + 3x + 2 = 0$.

10. $x^2 - 3x = 0$.

Discuss the results of the following problems:

11. Three boys out hunting shot altogether 21 rabbits. Two of the boys shot an equal number, but the third boy shot 2 more than either. How many rabbits did each shoot?

12. Three boys gathering nuts get altogether 21 bushels. Two of the boys gather the same amount, but the third boy gathers 2 bushels more than either of the others. How many bushels did each gather?

66. Negative numbers. The numbers consisting of positive integers and fractions are sometimes called the numbers of arithmetic. With this set of numbers, the operations of addition, subtraction, multiplication, and division, can be performed with but one exception. This exception is the subtraction of a larger number from a smaller, which is declared impossible. For example, there is no number in the set which answers the question, "What number added to 4 gives 3." If a student, knowing only this set of numbers, knows how to solve an equation, he would declare the solution of the equation $2x + 3 = 0$ to be impossible, or he might say the answer is imaginary.

In reading a thermometer, in debits and credits, in latitude and longitude, a physical meaning is found for the subtraction of a larger number from a smaller. The introduction of new numbers called negative numbers is one of the purposes of the study of algebra. With positive and negative integers and fractions the student is enabled to solve any problem which reduces to an equation of the first degree. The equation

$$ax + b = 0$$

of Art. 22 has always a solution among these numbers.

EXERCISES

Assuming that we know only positive numbers, show which of the following exercises are possible of solution.

1. $2x - 1 = 0$.
2. $3x - 8 = 0$.
3. $x + 2 = 4$.
4. $4x + 9 = 1$.
5. $5x - 2 = x$.
6. $x^2 - 5x + 6 = 0$.
7. $x^2 - x - 2 = 0$.
8. $x^2 - 3x + 2 = 0$.
9. $2x^2 + 3x - 2 = 0$.
10. $x^2 + 1 = 0$.

How many answers are there to each of the following problems?

11. Two thermometers differ by 3 degrees in reading. If the reading of one thermometer is doubled it equals the reading of the other. What are the readings of the two thermometers?

12. The ages of two children differ by 3 years. The age of one child is double the age of the other. What are the ages of the children?

67. Irrational numbers. The quotient of two integers is called a **rational** number. Since an integer may be written as a fraction with 1 as a denominator, the numbers discussed in Arts. 65 and 66 are rational numbers.

For example, 3, -6, $\frac{1}{3}$, .67, $-\frac{5}{2}$ are rational numbers.

In operations with radicals we often deal with **irrational** numbers. Irrational numbers cannot be expressed as the quotient of two integers.

For example, $\sqrt{2}$, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{4}$ are irrational.

A student having only rational numbers at his command would find it impossible to obtain the length of a diagonal of a unit square. In other words, he would not be able to solve the equation $x^2 = 2$. A further extension of the number system to include irrationals is then necessary.

EXERCISES

Represent by points on a straight line the following numbers:

1. $\frac{1}{2}$, -3, $-\frac{2}{3}$, $\frac{7}{2}$, 0.

2. $\sqrt{2}$.

SOLUTION: Construct a square of side 1 (Fig. 30). The length of its diagonal is $\sqrt{2}$. Laying off the length of this diagonal from 0 we find a point on the line representing $\sqrt{2}$.

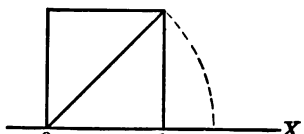

FIG. 30

3. $\sqrt{8}$.

4. $2 + \sqrt{2}$, $2 - \sqrt{2}$, $-2 - \sqrt{2}$, $-2 + \sqrt{2}$.

5. $3 + \sqrt{8}$, $-3 + \sqrt{8}$.

Arrange the following sets of numbers in ascending order.

6. $\sqrt{2}$, $-\frac{1}{2}$, $2 - \sqrt{2}$, $-\sqrt{8}$, 3, $2\sqrt{2}$.

7. 0, $-\frac{3}{2}$, $\frac{3}{2}$, $-\sqrt{2}$, -2, $\frac{5}{2}$, $-3\sqrt{2}$.

68. Real numbers. The four classes of numbers which we have discussed, that is, positive integers, fractions, irrationals and negative numbers, together with zero are classed together under the name **real numbers.** These numbers have the important property that they may all be represented by points on the same straight line.

69. Imaginaries. We have seen that man found it desirable to invent fractions, irrationals and negative numbers. In just the same way it is found convenient to invent still another kind of number. The square of any real number, positive or negative, is a positive number. The square root of a negative number cannot then be a real number and is given the name **imaginary number.** It is not represented on the same line with the real numbers.

The term "imaginary numbers" is here used in a technical sense. The numbers are imaginary in the same sense that a

fraction, a negative number or an irrational number is imaginary for a person knowing only the positive integers.

It is desirable to include these imaginary numbers in our number system; for otherwise we could not solve such an equation as

$$x^2 + 1 = 0.$$

The number $\sqrt{-1}$ bears somewhat the same relation to the system of imaginary numbers that the number 1 does to the real numbers. Any real number may be written with 1 as a coefficient; for example, $4 = 1 \cdot 4$. Any imaginary number involves the product of a real number and $\sqrt{-1}$.

Thus, we may write

$$\sqrt{-4} = 2\sqrt{-1},$$
$$\sqrt{-5} = \sqrt{5}\,\sqrt{-1},$$
$$\sqrt{-\tfrac{1}{2}} = \sqrt{\tfrac{1}{2}}\,\sqrt{-1},$$
$$2 + \sqrt{-9} = 2 + 3\sqrt{-1}.$$

The number $\sqrt{-1}$ occurs so often that a special symbol is used for it; that is,

$$\sqrt{-1} = i.$$

EXERCISES

Write the following imaginary numbers in terms of $i = \sqrt{-1}$:

1. $\sqrt{-5}$.

SOLUTION: $\sqrt{-5} = \sqrt{5 \cdot -1} = \sqrt{5} \cdot \sqrt{-1} = i\sqrt{5}$.

2. $\sqrt{-10}$.
3. $\sqrt{-9}$.
4. $\sqrt{-\frac{3}{4}}$.
5. $\sqrt{-a}$.
6. $\sqrt{-x^2}$.
7. $\sqrt{-x^2 - y^2}$.
8. $\sqrt{\dfrac{-x^2}{y^2}}$.
9. $\sqrt{-9x^2 - 9}$.
10. $\sqrt{-1 - 4x^2}$.

70. Powers of i. Since any imaginary may be written in terms of i, it is important to consider the powers of i. We operate with i as with any other letter and remember that i^2 is to be replaced by -1. Hence, for powers of i, we have

$$\begin{aligned} i^2 &= -1, \\ i^3 &= i^2 \cdot i = -i, \\ i^4 &= i^2 \cdot i^2 = (-1)(-1) = 1, \\ i^5 &= i^4 \cdot i = i, \\ i^6 &= i^4 \cdot i^2 = -1, \end{aligned}$$

and so on, repeating the numbers -1, $-i$, $+1$, $+i$.

71. Products of imaginaries. If imaginary numbers are written in terms of i, the product of any number of imaginaries is easily found.

Example. Find the product of $\sqrt{-3}$, $\sqrt{-5}$, and $\sqrt{-7}$.

SOLUTION: $\sqrt{-3} = i\sqrt{3}$, $\sqrt{-5} = i\sqrt{5}$, $\sqrt{-7} = i\sqrt{7}$.

$$\begin{aligned} \sqrt{-3} \cdot \sqrt{-5} \cdot \sqrt{-7} &= i\sqrt{3} \cdot i\sqrt{5} \cdot i\sqrt{7} \\ &= i^3\sqrt{3} \cdot \sqrt{5} \cdot \sqrt{7} \\ &= -i\sqrt{105}. \end{aligned}$$

In general if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries we have

$$\sqrt{-a} \cdot \sqrt{-b} = i\sqrt{a} \cdot i\sqrt{b} = i^2\sqrt{a} \cdot \sqrt{b} = -\sqrt{a} \cdot \sqrt{b} = -\sqrt{ab},$$

or

$$\sqrt{-a} \cdot \sqrt{-b} = -\sqrt{ab}.$$

EXERCISES

1. Continue the table of Art. 70 for i^7, i^8, i^9, i^{10}, i^{11}.

Find the following products:

2. $\sqrt{-2} \cdot \sqrt{-3}$.
3. $\sqrt{-3} \cdot \sqrt{-5}$.
4. $\sqrt{-9} \cdot \sqrt{-10}$.
5. $\sqrt{-4} \cdot \sqrt{-16}$.
6. $\sqrt{-2} \cdot \sqrt{-6} \cdot \sqrt{-11}$.
7. $\sqrt{-2} \cdot \sqrt{-2} \cdot \sqrt{-3}$.
8. $(\sqrt{-2})^3$.
9. $(\sqrt{-5})^4$.
10. $(\sqrt{-4})^5$.
11. $(\sqrt{-3})^3\sqrt{-4}$.
12. $(\sqrt{-\frac{1}{2}})^4\sqrt{-\frac{1}{3}}$.

72. Complex numbers. The sum of a real number and an imaginary number is called a **complex number.** Thus, $1 + i$, $2 - \sqrt{-3}$, $5 + 3i$ are complex numbers. In general, a complex number has the form

$$a + ib,$$

where a and b are real numbers. The expression "imaginary numbers" is often used to mean complex numbers, and the term **pure imaginary** is applied to numbers of the form ib.

We shall find such numbers in solving quadratic equations.

For example, $2 + \sqrt{-3}$ satisfies the equation

$$x^2 - 4x + 7 = 0,$$

as shown by actual substitution.

Thus,

$$(2 + \sqrt{-3})^2 - 4(2 + \sqrt{-3}) + 7 = 4 + 4\sqrt{-3} - 3 - 8 - 4\sqrt{-3} + 7 = 0.$$

73. Operations with complex numbers. The ordinary operations of algebra may be performed with complex numbers by operating with i as with any other letter and remembering that $i^2 = -1$.

Example 1. Add $6 + \sqrt{-3}$ and $3 - \sqrt{-2}$.

In the i notation we are to find

$$(6 + i\sqrt{3}) + (3 - i\sqrt{2}) = 6 + i\sqrt{3} + 3 - i\sqrt{2} = 9 + i(\sqrt{3} - \sqrt{2}).$$

Example 2. Multiply $6 + \sqrt{-3}$ by $3 - \sqrt{-2}$. In the i notation

$$(6 + i\sqrt{3})(3 - i\sqrt{2}) = 18 - 6i\sqrt{2} + 3i\sqrt{3} - i^2\sqrt{3}\sqrt{2}$$

$$= 18 - i(6\sqrt{2} - 3\sqrt{3}) - (-1)\sqrt{6} = 18 + \sqrt{6} - i(6\sqrt{2} - 3\sqrt{3}).$$

74. Conjugate complex numbers. Complex numbers which differ only in the sign of the imaginary parts are called **conjugate numbers.** Thus, $3 + 2i$ and $3 - 2i$, $a + ib$ and $a - ib$ are conjugate. Since

$$(a + ib) + (a - ib) = 2a,$$
$$(a + ib) - (a - ib) = 2ib,$$
$$(a + ib)(a - ib) = a^2 + b^2,$$

we see that the sum and product of two conjugate complex numbers are real numbers, but the difference is an imaginary number.

EXERCISES

Perform the following operations:

1. $(2 - 3i) + (3 + 4i)$.
2. $(1 - 2i) + (5 - 7i) + (2 - i)$.
3. $(5 + 8i) - (2 + 5i)$.
4. $(6 + \sqrt{-3}) + (5 - \sqrt{-2})$.
5. $(6 + \sqrt{-3}) - (5 - \sqrt{-2})$.
6. $(6 + \sqrt{-3})(5 - \sqrt{-2})$.
7. $(3 - 4i)(3 + 4i)$.
8. $(3 - 4i)^2$.
9. $(5 - 2i)(3 + i) - (1 + i)$.
10. $(2 + \sqrt{-4})(1 - \sqrt{-1})(3 + \sqrt{-16})$.
11. $(1 - \sqrt{-1})(1 + \sqrt{-1}) + (2 - 3\sqrt{-1})(2 + 3\sqrt{-1})$.
12. $(1 - i)^2$.
13. $(1 - \sqrt{-1})^3$.
14. $(2 + \sqrt{-25})^2$.
15. $(3 + 2i) \div (2 + 4i)$.

SOLUTION: We are to find a complex number equal to $\frac{3 + 2i}{2 + 4i}$. Multiply the numerator and denominator by the conjugate of the denominator.

$$\frac{3 + 2i}{2 + 4i} = \frac{(3 + 2i)(2 - 4i)}{(2 + 4i)(2 - 4i)} = \frac{14 - 8i}{20} = \frac{7}{10} - \frac{2i}{5}.$$

16. $(3 + 2i) \div (1 + i)$.
17. $(4 - i) \div (2 - 3i)$.
18. $(1 + \sqrt{-1}) \div (1 - \sqrt{-1})$.
19. $(3 + \sqrt{-2}) \div (4 - \sqrt{-3})$.
20. $(1 + i)^2 + (2 - i)^2$.
21. $(1 + i)^2 - (2 - i)^2$.
22. $(1 + i)^2 \cdot (2 - i)^2$.
23. $(1 + i)^2 \div (2 - i)^2$.
24. $[(1 + \sqrt{-2}) + (3 - 4\sqrt{-1})]\ [1 + 2\sqrt{-1}\ ]^2$.

***75. Graph of a pure imaginary number.** We have shown that any real number a may be represented on a straight line.

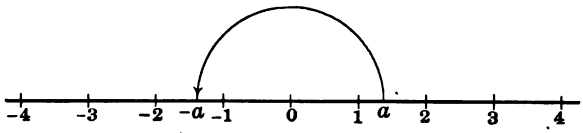

FIG. 31

If we multiply a by -1, we get $-a$ which is represented by a point a distance a to the left of O. (Fig. 31.) We may then look upon the multiplication by -1 as an operation which turns the line Oa about O through an angle of 180°. Since $-1 = i \cdot i$, we may consider multiplication by i as an operation which must be performed twice to give a turn of 180°. That is, we may look upon the multiplication by i as an operation which turns the line Oa through a right angle. The number ai is then represented by a point on the vertical line at a distance a upward from the horizontal line. (Fig. 32.)

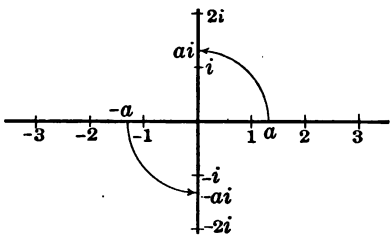

FIG. 32

The symbol i may be used to denote direction just as we use + or −. Thus, +2 means a distance 2 to the right, −2, a distance 2 to the left, $2i$, a distance 2 up, $-2i$, a distance 2 down, all measured from O. (Fig. 33.)

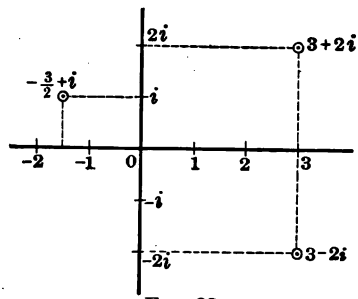

FIG. 33

***76. Graph of a complex number.** The complex number may be graphically represented by

* These articles may be omitted without breaking the continuity of the course.

a point in a plane by using the symbol i as a symbol of direction. Thus $3 + 2i$ may be read, "a distance 3 to the right, then a distance 2 up." The point in the plane thus designated represents this complex number and no other number.

In the same way, $3 - 2i$ reads "a distance 3 to the right then a distance 2 down." Any complex number may be represented by a point in the plane, and any point in the plane is the graph of some complex number. (Fig. 33.)

EXERCISES

Represent the following complex numbers graphically:

1. $2 + 2i$, $-2 + 2i$, $-2 - 2i$, $2 - 2i$.

2. $1 - i$, $3 - i$, $3 + 5i$, $-3 + 5i$, $-1 - 4i$.

3. 1, i, $2i$, $3 + \sqrt{-2}$, $6 + \frac{5}{2}\sqrt{-1}$.

4. Draw the triangle whose vertices are $2 - 3i$, $3 + 2i$, and $1 - i$.

5. Represent graphically, $3 + 4i$, $3 - 4i$, and their sum.

6. Show that the sum of two conjugate complex numbers is always represented by a point on the horizontal line through O. (Fig. 33.)

7. Represent graphically, $5 - 2i$, $5 + 2i$, and their difference.

8. Show that the difference of two conjugate numbers is always represented by a point on the vertical line through O.

9. How can a real number be considered as included among the numbers of the form $a + ib$?

10. How can a pure imaginary number be included among the numbers of the form $a + ib$?

Perform the following operations:

11. $(3 - 2\sqrt{-5}) - (2 - 3\sqrt{-5})$.

12. $(3 - \sqrt{-10}) + (4 - 2\sqrt{-10})$.

13. $(2\sqrt{-3})\ (1 + \sqrt{-1})$.

14. $(1 + \sqrt{-2})\ (1 - \sqrt{-2})\ (2 + \sqrt{-1})$.

15. $(1+\sqrt{-3})^3$.

16. $(\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3$.

17. $3 \div (1-\sqrt{-1})$.

18. $\dfrac{8+i}{3+2i}$.

19. Show by actual substitution that $2-\sqrt{-3}$ is a solution of the equation $x^2 - 4x + 7 = 0$.

20. By actual substitution show that $-1+i$ and $-1-i$ are solutions of $x^2 + 2x + 2 = 0$.

21. Show that $-1+\sqrt{-3}$ is a solution of the equation $x^3 - 8 = 0$.

Historical note on imaginary numbers. It appears that Cardan (1501–1576), the famous algebraist of Milan, took notice of negative roots of an equation, but called such roots fictitious. An equation whose roots are imaginary was still treated by Cardan as impossible of solution. But soon after the time of Cardan, a Flemish mathematician, Girard (1590–1634), mentioned imaginary numbers. The use of imaginaries developed slowly, and it was in 1800 when Wessel, a Norwegian, and Argand, a Frenchman, gave such a graphical representation of imaginaries as is shown in Fig. 33. It was really in the work of Gauss (1777–1855), the great German mathematician, that such systematic use of imaginary numbers was made as to give a full and general understanding of their importance in the theory of algebra.

REVIEW EXERCISES AND PROBLEMS ON CHAPTERS V–VIII

1. If x varies inversely as y, and when $x = 6$, $y = 2$, what is the value of x for $y = 3$?

2. If x varies as y, show that $x + y$ varies as y.

3. Determine x and y from

$$0.5x - 3y = 0.15,$$
$$4x + 0.2y = 2.41.$$ (STANFORD) *

4. Solve

$$\frac{2}{x} + \frac{3}{y} = 4,$$
$$\frac{1}{x} + \frac{7}{y} = 6.$$ (YALE)

5. Find a mean proportional between $(x + y)^2$ and $(x - y)^2$.

6. Find a third proportional to $\frac{a}{b} + \frac{b}{a}$ and $\frac{a}{b}$.

7. Evaluate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}.$$

8. Solve the equation

$$\begin{vmatrix} x & 2 & 5 \\ x & 3 & 6 \\ 1 & 4 & 7 \end{vmatrix} = 0.$$

9. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs one pound at the surface of the earth. What would it weigh 1500 miles above the surface? (Take 4000 miles for the radius of the earth.)

10. The weight of an object below the surface of the earth varies directly as its distance from the center of the earth. An object weighs one pound at the surface of the earth. What would it weigh at a point 500 miles below the surface?

11. Simplify $(a - ib)^2 - (a + ib)^2$.

12. Represent graphically on the same diagram the numbers $2 - 3i$, $3 - \sqrt{-1}$, i, $2i$, 3, $\frac{3i}{2}$, $\frac{1}{2} - 5i$, $\sqrt{2} - i\sqrt{2}$.

13. Solve

$$3ax - 2by = c,$$
$$a^2x + b^2y = 5bc.$$ (SHEFFIELD)

14. If the base of a triangle is constant, show that the area varies directly as the altitude.

* Institution that gave the question in an entrance examination.

15. Show that the radius of a circular cylinder varies inversely as the square root of the altitude if the volume is constant.

16. The price of sugar for the 21 days immediately following the opening of the European war was as follows: 4.40, 4.40, 4.40, 4.40, 5.00, 5.00, 5.00, 6.00, 6.50, 6.75, 7.50, 7.50, 7.50, 7.50, 7.50, 7.50, 7.25, 7.25, 7.00, 7.00, 7.00. Show the change in price by a graph.

17. On the same sheet of coördinate paper, represent graphically the three equations $y = x^2 + x + 1$, $y = 1 - x$, $y = \frac{2}{x} + 1$. What can be told from the points where the graphs intersect each other?

18. Give an example of two linear equations in two unknowns whose graphs are parallel.

19. Give an example of two linear equations whose graphs coincide.

20. If $a : b = c : d$, prove that $ma + b : ma - b = mc + d : mc - d$.

21. If $7x - 4z : 8x - 3z = 4y - 7z : 3y - 8z$, prove that z is a mean proportional between x and y.

22. The following scores were made by city school children and by rural school children in grades from the third to the eighth inclusive, in a speed test in the fundamental operations in arithmetic.

Grade	3rd	4th	5th	6th	7th	8th
City	5.4	6.6	9.0	10.3	11.5	13.1
Rural	2.5	6.5	6.9	8.0	10.0	11.1

Make graphs of these scores to show the comparative speeds.

23. Solve:
$$\frac{1}{x} + \frac{1}{y} = 1,$$
$$\frac{1}{y} + \frac{1}{z} = \frac{3}{2},$$
$$\frac{1}{z} + \frac{1}{x} = 2.$$
(MASS. INSTITUTE)

24. If one of two numbers be multiplied by m and the other by n, the sum of the products is P, but if the first be multiplied by m' and the second by n', the sum of the products is P'. Find the numbers. (DARTMOUTH)

25. Represent graphically, $y = x^2 - 2x$. (WISCONSIN)

26. Compare the cubes of the following numbers,
$$1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}.$$

27. By actual substitution show that $3 + 2i$ and $3 - 2i$ satisfy the equation $x^3 - 6x^2 + 13x = 0$.

28. Given that z varies as the sum of two quantities one of which varies directly as x and the other as x^2. If when $x = 1, 2, 3$, $z = 5, 16, 33$ respectively, what is z in terms of x?

29. Simplify $(i^9 + i^{10} + i^{11} + i^{12})^7$, if $i^2 = -1$.

30. Find the value of y by determinants from the equations:

$$3x - 4y + 2z = 1,$$
$$2x + 3y - 3z = -1,$$
$$5x - 5y + 4z = 7.$$

(COLLEGE ENTRANCE EXAMINATION BOARD)

31. Simplify $(3\sqrt{-2} + \sqrt{-4})(2\sqrt{-3} - \sqrt{-2})$; $\dfrac{\sqrt{-2} + 3\sqrt{-1}}{\sqrt{-2} - \sqrt{-1}}$.

(ILLINOIS)

32. The force P necessary to lift a weight W by means of a certain machine is given by the formula $P = a + bW$, where a and b are constants depending on the amount of friction in the machine. If a force of 7 pounds will raise a weight of 20 pounds, and a force of 13 pounds will raise a weight of 50 pounds, find the force necessary to raise a weight of 40 pounds. (First determine the constants a and b.) (HARVARD)

33. Solve the simultaneous equations

$$\begin{vmatrix} x & y & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = 2, \quad \begin{vmatrix} x & 1 & y \\ 1 & -1 & 1 \\ -1 & 2 & 0 \end{vmatrix} = 0.$$

34. If $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$, prove that $ax + by + cz$ is a mean proportional between $x^2 + y^2 + z^2$ and $a^2 + b^2 + c^2$.

35. Find a number such that by adding it to each of the numbers, a, b, c, d, we obtain four numbers in proportion. (MASS. INSTITUTE)

36. A glass of milk and cream is sold for the same price as a glass of milk of twice the size. If milk costs 7 cents a quart and cream 32 cents a quart, what is the percentage of cream in the smaller glass? (HARVARD)

37. Solve:

$$4x - 8y = -3,$$
$$11x + 5y = -15.$$

Draw a graph and verify your answer. (MISSOURI)

38. If $a : b = c : d$, prove that $ab + cd$ is a mean proportional between $a^2 + c^2$ and $b^2 + d^2$. (MASS. INSTITUTE)

39. A man invested \$2720 in railroad stock, a part at 95 yielding 2 per cent, and the balance at 82 yielding 3 per cent; his income from both investments is \$70. Find the amount invested in each kind of stock.

(ILLINOIS)

40. Suppose the earth to be a smooth sphere 25,000 miles in circumference and that an iron band is fitted closely around the earth at the equator. Suppose this band to be expanded by the heat of the sun until its inner circumference is 25,000 miles and 1 foot in length, and that the distance between the surface of the earth and the band is the same at all points. Is it possible to insert a knife blade $\frac{1}{16}$ inch thick between the surface of the earth and the band? Estimate your answer and then check it.

41. A concrete pavement is 28 feet wide. A cross section gives an arc of a circle with an 8-inch rise at the center. (Fig. 34.) It is necessary to determine the radius of this circle in order to construct the form (templet) by which the concrete is laid. Find the radius.

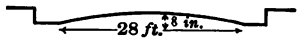

FIG. 34

42. Solve the above problem if the width of the pavement is a and the rise at the center is b.

43. The lengths of the sides of a triangle are consecutive integers. If a represents the side of middle length and A the area of the triangle, then $A = f(a)$. Find $f(a)$.

44. If 8 yards of silk and 12 yards of woolen cost \$27, and 12 yards of silk and 8 yards of woolen cost \$28, find the price per yard of the silk and of the woolen. (CALIFORNIA)

45. The radius r of one base of a segment of a sphere is one-half the radius of the other base. The altitude of the segment is equal to the radius of the smaller base. The volume of the segment and the surface of the zone are then functions of r. Find these functions. (See Formulas 18 and 19, Art. 45.)

CHAPTER IX

QUADRATIC EQUATIONS

77. Typical form. A **quadratic** equation in one unknown x is an equation that can, by transposing and collecting terms, be written in the typical form

$$ax^2 + bx + c = 0,$$

where a, b, c do not involve x, and have any values with the one exception that a is not zero. Since the result of multiplying the members of an equation in this typical form by any given number is an equation in the typical form, the a, b, c can be selected in an indefinitely large number of ways.

A quadratic equation is said to be of the second degree.

The function $ax^2 + bx + c$ $(a \neq 0)$ is called the **typical quadratic function.**

EXERCISES

Arrange the following equations in the typical form and select a, b, and c from the resulting equations:

1. $3x^2 + 4 + \frac{x}{2} = \frac{x^2}{3} + x + 1.$

SOLUTION: By transposing and collecting terms,

$$\frac{8}{3}x^2 - \frac{x}{2} + 3 = 0,$$

so that $$a = \frac{8}{3}, b = -\frac{1}{2}, c = 3.$$

2. $3 + x - 7x^2 = 2x + 9 - 9x^2.$

3. $(x + 1)^2 = x + 2.$

4. $\frac{x^2}{2} + \frac{x}{3} + \frac{1}{4} = 2x^2 + 3x - 4.$

5. $4y^2 + \frac{1}{3}y + 6 = 4 - 3y + 5y^2.$

6. $x^2 + 3x - x + 4 = 2x + 1.$

7. $\frac{x^2}{2} + \frac{x}{5} + 1 + x^2 - \frac{2x}{15} = 1 + \frac{x}{15}.$

8. $3x^2 - x + k = \frac{2}{3}x^2 + 2.$

SOLUTION: By transposing and collecting terms,

$$\frac{7}{3}x^2 - x + k - 2 = 0,$$

so that $$a = \frac{7}{3}, b = -1, c = k - 2.$$

9. $x^2 + (2x + n)^2 = 1.$

10. $x^2 + 2nx + t = k + tx.$

11. $(mx + n)^2 + x^2 = n^2.$

12. $\frac{(x + 1)^2}{4} + \frac{(x - 2)^2}{9} = r^2.$

13. $(k - x)^2 - (n + x)^2 = (m - x)^2.$

14. $d^2y^2 + y^2 + \frac{d}{2} + (y + d)^2 = 0.$

15. $(z + n)^3 - (z - n)^3 = n^3.$

78. Solution by factoring. In solving a quadratic equation, it is generally best first to reduce the equation to the typical form. After such a reduction, if the left-hand member of the equation can be factored readily, the solution is very simple.

Example. Solve $(x - 3)^2 = 6 - 2x.$

Arranged in the typical form this equation becomes

$$x^2 - 4x + 3 = 0.$$

The factors of the left-hand member are easily found. They are $x - 3$ and $x - 1$, and we may write the equation in the form

$$(x - 3)(x - 1) = 0.$$

Any value of x which makes either factor zero will satisfy the equation.

If $x = 3$, we have $(3 - 3)(3 - 1) = 0 \cdot 2 = 0.$

Again, if $x = 1$, we have $(1 - 3)(1 - 1) = -2 \cdot 0 = 0.$

Hence, 3 and 1 are solutions of the given quadratic equation.

EXERCISES

Solve the following by factoring:

1. $x^2 + 6x + 5 = 0$.
2. $(x + 3)^2 = 1$.
3. $x^2 = 625$.
4. $x^2 = 7x$.
5. $(x + 1)(x - 1) - 8 = 0$.
6. $x^2 - nx = mn - mx$.
7. $x^2 - 2nx + n^2 = 0$.
8. $2x^2 + 5x - 3 = 0$.
9. $3x^2 + 4x + 1 = 0$.
10. $3x^2 - 17x + 10 = 0$.
11. $6y^2 - 5y + 2 = 1$.
12. $2s^2 - 3s = 9$.
13. $4 + z(4z - 17) = 0$.
14. $2x^2 - ax - a^2 = 0$.
15. $(x + 1)^3 = x(x^2 - 6x - 3)$.
16. $a^3 = 3a^2 - 2a$.

79. Solution by formula. Any quadratic equation may be solved by the process of "completing the square."

For example, to solve $2x^2 - 4x - 7 = 0$, write the equation in the form $x^2 - 2x = \frac{7}{2}$. Adding 1 to both members to make the left-hand member a perfect square,

$$x^2 - 2x + 1 = \frac{7}{2} + 1 = \frac{9}{2},$$

or

$$(x - 1)^2 = \frac{9}{2}.$$

Extract the square root,

$$x - 1 = \pm\sqrt{\frac{9}{2}} = \pm\frac{3}{2}\sqrt{2},$$

and

$$x = 1 + \frac{3}{2}\sqrt{2}, \text{ or } 1 - \frac{3}{2}\sqrt{2}.$$

Both of these values of x satisfy the original equation, as we see on substituting them for x. Thus,

$$2\left(1 + \frac{3}{2}\sqrt{2}\right)^2 - 4\left(1 + \frac{3}{2}\sqrt{2}\right) - 7 = 2\left(1 + 3\sqrt{2} + \frac{9}{4} \cdot 2\right) - 4\left(1 + \frac{3}{2}\sqrt{2}\right) - 7.$$

$$= 2 + 6\sqrt{2} + 9 - 4 - 6\sqrt{2} - 7 = 0.$$

In the same way, substituting $1 - \frac{3}{2}\sqrt{2}$ for x, we find

$$2(1 - \tfrac{3}{2}\sqrt{2})^2 - 4(1 - \tfrac{3}{2}\sqrt{2}) - 7 = 0.$$

If we apply this method to the typical quadratic equation

$$ax^2 + bx + c = 0,$$

we obtain a formula by means of which we can write down the roots of an equation without factoring or completing the square.

Transposing c and dividing by a, the general equation becomes

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

Add the square of one half the coefficient of x to each member to make the left-hand member a perfect square,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

or

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

If we let x_1 and x_2 represent the two solutions of the quadratic equation

$$ax^2 + bx + c = 0,$$

we may write

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

These expressions may be used as formulas for the solution of any quadratic equation.

Thus, to solve the equation

$$2x^2 - 4x - 7 = 0,$$

we substitute in the formulas, $a = 2$, $b = -4$, $c = -7$, and find

$$x_1 = \frac{4 + \sqrt{16 - 4 \cdot 2 \cdot (-7)}}{4} = \frac{4 + \sqrt{72}}{4} = \frac{2 + 3\sqrt{2}}{2} = 1 + \frac{3}{2}\sqrt{2},$$

$$x_2 = \frac{4 - \sqrt{16 - 4 \cdot 2 \cdot (-7)}}{4} = 1 - \frac{3}{2}\sqrt{2}.$$

EXERCISES

Solve the following equations by use of the formulas, and verify by substitution:

1. $x^2 - 4x + 3 = 0.$
2. $x^2 - 7x + 10 = 0.$
3. $x^2 + 7x + 12 = 0.$
4. $3x^2 + 10x = 32.$
5. $2x^2 - 3x - 14 = 0.$
6. $4x^2 + 7x + 3 = 0.$
7. $3x^2 - 7x - 6 = 0.$
8. $6x^2 + 5x + 1 = 0.$
9. $6x^2 + x - 12 = 0.$
10. $9x^2 = 13 - 4x.$
11. $9x^2 + 14x + 3 = 0.$
12. $3x^2 - 18x + 5 = 0.$
13. $10x^2 + 9x + 8 = 0.$
14. $x^2 + 6.51 - 5.2x = 0.$
15. $(1 - x)(x - 2) + 3 = 0.$
16. $x^2 + \frac{1}{2}x - \frac{3}{16} = 0.$
17. $y^2 + 0.9y = 1.2.$

Solve by any method:

18. $16(1 - x)(1 + x) - 9x^2 = 0.$
19. $x^2 + 9x + 14 = 0.$
20. $x^2 + 7x - 9 = 0.$
21. $.3x^2 + 1.03x + .1 = 0.$
22. $x(x - 2) = 5 + 3x.$
23. $(2x - 5)^2 - (x - 6)^2 = 80.$
24. $3x^2 = 33x - 90.$
25. $(x - 6)(x - 5) + (x - 7)(x - 4) = 10.$

26. Divide the number 29 into two parts such that their product is 198.

27. A square is cut from a rectangular piece of paper. The remainder of the rectangle has the shape and dimensions shown

in Fig. 35. What is the side of the square if the area of the original rectangle was 100 square inches.

28. The product of two consecutive numbers is 182. Find the numbers.

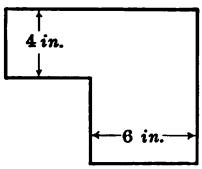

FIG. 35

29. Three times the product of two consecutive even integers exceeds ten times their sum by 60. Find the numbers.

30. The sum of two adjacent sides of a rectangle is 28 inches, and the area of the rectangle is 195 square inches. What are the dimensions of the rectangle?

31. One side of a rectangle is 3.5 inches longer than the other. The area of the rectangle is 181 square inches. What are its dimensions?

32. A picture whose dimensions are 56 x 60 inches has a frame of uniform width whose area equals that of the picture. Find the width of the frame.

33. Within a rectangle whose sides are 30 and 40 inches, a second rectangle is drawn so that its sides are everywhere equally distant from the sides of the given rectangle. The area of the second rectangle is one-half that of the first. Find the perimeter of the second rectangle.

80. Special or incomplete quadratics. If b or c is zero in the quadratic equation

$$ax^2 + bx + c = 0,$$

the equation is said to be incomplete.

I. When $c = 0$, $ax^2 + bx = 0$ is the typical form of the equation. We can always write this equation in the form

$$x(ax + b) = 0.$$

Hence, the roots are 0 and $\frac{-b}{a}$. Conversely, if 0 is a root of a quadratic equation

$$ax^2 + bx + c = 0,$$

then $a \cdot 0 + b \cdot 0 + c = 0.$

That is, $c = 0.$

Therefore, *a quadratic equation has a root equal to zero when and only when the equation has no known term.*

II. When $b = 0$, $ax^2 + c = 0$ is the typical form.

In this case, $x = \pm\sqrt{\frac{-c}{a}}.$

Conversely, if the roots of a quadratic equation are arithmetically equal, but opposite in sign, there is no term containing x in the first degree; for if $+r$ and $-r$ are both roots of

$$ax^2 + bx + c = 0, \quad (1)$$

we have $$ar^2 + br + c = 0, \quad (2)$$

and $$ar^2 - br + c = 0. \quad (3)$$

Subtracting (3) from (2), $2br = 0.$

Since it is given that r is not zero, it follows that $b = 0$.

Hence, *a quadratic equation has two roots arithmetically equal but opposite in sign, when and only when the term in x vanishes.*

III. When $b = 0$, $c = 0$, the typical form is $ax^2 = 0$. *Both roots of a quadratic equation are equal to zero when and only when the known term and the term in x vanish.*

Solve: **EXERCISES**

1. $(x - 6)^2 = 72 - 12x.$ **3.** $ax^2 - b = cx^2 + d.$

2. $\frac{x^2}{9} = \frac{(x + 3)(x - 3)}{8}.$ **4.** $x^2 - 3x = 0.$

5. $(x + 1)^2 + (x + 2)(x + 3) = 7.$

6. $(5 - x)(x - 6) + 30 = 0.$

Determine k so that each of the following equations shall have one root equal to zero:

7. $3x^2 + 8x + 3k - 15 = 0.$ **8.** $x^2 + 7x + 11 - k^2 = 0.$

9. $2y^2 - \frac{y}{2} + k - 4k^2 = 0.$

Determine k and m so that each of the following equations shall have two roots equal to zero:

10. $5x^2 + 3kx - 2mx - 5x + k + m - 5 = 0.$

11. $2x^2 + 4k^2x + 3kx - 3m - 1 = 0.$

Determine k so that the roots of the following equations may be arithmetically equal but opposite in sign:

12. $2x^2 + x + 2kx + 1 = 0.$

13. $k^2x - 9x - 3x^2 = 14.$

14. $z^2 + 7k^2z - 2kz - 8 = 0.$

81. Formation of equations with given roots. It is sometimes necessary to form a quadratic equation with given roots. The following theorem and its converse enable us to do this very easily.

THEOREM. *If r is a root of the equation*

$$ax^2 + bx + c = 0,$$

then $x - r$ is a factor of the left-hand member.

If r is a root of the equation, we have

$$ar^2 + br + c = 0.$$

Then
$$\begin{aligned} ax^2 + bx + c &= ax^2 + bx + c - (ar^2 + br + c) \\ &= a(x^2 - r^2) + b(x - r) \\ &= (x - r)(ax + ar + b). \end{aligned}$$

Hence, $x - r$ is a factor of $ax^2 + bx + c$.

CONVERSE THEOREM. *If $x - r$ is a factor of the left-hand member, then r is a root of the equation.*

If $x - r$ is a factor of $ax^2 + bx + c$, then the substitution of r for x makes the factor $x - r$ vanish, and r is a root of

$$ax^2 + bx + c = 0.$$

EXERCISES

Form quadratic equations of which the following are roots:

1. $3, \frac{1}{2}$.

SOLUTION: When the right-hand member of the equation to be formed is 0, the left-hand member has factors $x - 3$ and $x - \frac{1}{2}$. Hence

$$(x - 3)(x - \tfrac{1}{2}) = x^2 - \frac{7}{2}x + \frac{3}{2} = 0$$

is a quadratic equation with roots 3 and $\frac{1}{2}$. Since we may multiply both members of the equation by 2 we may write the equation freed from fractional coefficients in the form

$$2x^2 - 7x + 3 = 0.$$

2. $2, \frac{1}{2}$.

3. $3, \frac{-1}{3}$.

4. $-4, -3$.

5. $\sqrt{3}, -\sqrt{3}$.

6. a, b.

7. $1 - \sqrt{3}, 1 + \sqrt{3}$.

8. $i, -i$.

9. $2 + 3i, 2 - 3i$.

10. $2 + 7\sqrt{3}, 2 - 7\sqrt{3}$.

11. $m + \frac{1}{m}, m - \frac{1}{m}$.

12. Form an equation whose roots are 2 and $\frac{2}{3}$, such that the coefficient of x^2 is 6.

13. Form equations whose roots are 2 and $\frac{2}{3}$ such that the coefficient of x is 5; such that the known term is 7.

82. Nature of the roots of a quadratic. In Art. 79, we found the two roots of the quadratic equation,

$$ax^2 + bx + c = 0,$$

to be $$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

and $$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

In case a, b, c are real numbers, the numerical character of these roots depends upon the number $b^2 - 4ac$ under the radical

sign. An examination of x_1 and x_2 leads at once to the following conclusions:

(1) *If* $b^2 - 4ac > 0$, *the roots are real and unequal.*

Thus, in the equation

$$3x^2 - 11x + 10 = 0,$$

we find $b^2 - 4ac = 11^2 - 4 \cdot 3 \cdot 10 = 1 > 0$, hence the roots of this equation are real and unequal. Upon solution they are found to be 2 and $\frac{5}{3}$.

(2) *If* $b^2 - 4ac < 0$, *the roots are imaginary and unequal.*

Thus, in the equation

$$x^2 - 6x + 58 = 0,$$

we find $b^2 - 4ac = 6^2 - 4 \cdot 58 = -196 < 0$, hence the roots of this equation are imaginary and unequal. Upon solution they are found to be $3 - 7i$ and $3 + 7i$.

(3) *If* $b^2 - 4ac = 0$, *the roots are real and equal.*

Thus, in the equation

$$4x^2 + 4x + 1 = 0,$$

we find $b^2 - 4ac = 4^2 - 4 \cdot 4 \cdot 1 = 0$, and the roots are real and equal. Upon solution they are found to be $-\frac{1}{2}$, $-\frac{1}{2}$.

It should be observed that if the coefficients are real and one root is imaginary, then both roots are imaginary.

The expression $b^2 - 4ac$ is called the **discriminant** of the equation

$$ax^2 + bx + c = 0.$$

If we add together the two roots of the equation

$$ax^2 + bx + c = 0,$$

we have

$$x_1 + x_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}.$$

Hence:

I. *The sum of the roots of a quadratic equation in* x *is equal to the coefficient of* x *with its sign changed, divided by the coefficient of* x^2.

Thus in the equation

$$3x^2 - 4x + 5 = 0,$$

we know immediately that the sum of the two roots is $\frac{4}{3}$.

If we multiply the two roots together, we have

$$x_1x_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) = \frac{c}{a}.$$

Hence:

II. *The product of the roots of a quadratic equation in x is equal to the known term divided by the coefficient of x^2.*

Thus, in the equation $3x^2 - 4x + 5 = 0$,

we know immediately that the product of the two roots is $\frac{5}{3}$.

EXERCISES

Determine the nature of the roots of the following equations:

1. $11x^2 - 23x + 2 = 0$.
2. $x^2 - 2x + 2 = 0$.
3. $2x^2 + 6x + 199 = 0$.
4. $49x^2 + 14x + 1 = 0$.
5. $y^2 + y + 1 = 0$.

Determine the real values of k so that the roots of the following equations may be equal:

6. $x^2 + x + k = 0$.

SOLUTION: In order that the roots of this equation may be equal we must have $b^2 - 4ac = 1 - 4k = 0$. Hence k is $\frac{1}{4}$. Substituting this value in the equation we have

$$x^2 + x + \tfrac{1}{4} = (x + \tfrac{1}{2})^2 = 0.$$

7. $x^2 + kx + 1 = 0$.
8. $2x^2 + 3x + k = 0$.
9. $x^2 + 3kx + k + 7 = 0$.
10. $kx^2 - 4x + 3 = 0$.

Determine by inspection the sum and the product of the roots of the following equations:

11. $4x^2 + 7x + 1 = 0$.
12. $y^2 + 3y - 11 = 0$.
13. $2 + 5x - 2x^2 = 0$.
14. $x^2 - 2x + 3k = 0$.
15. $x^2 + px + q = 0$.

Determine the value of k in the following equations:

16. $x^2 + kx + 3 = 0$, where one root is 2.

SOLUTION: Let x_1 be the second root. The product of the roots of this equation is 3.

Hence, $2 \cdot x_1 = 3$, or $x_1 = \frac{3}{2}$.

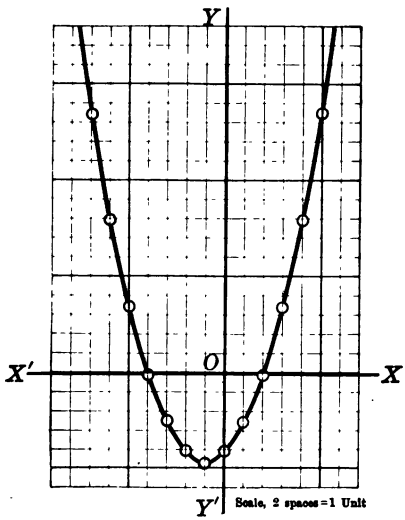

FIG. 36

The sum of the roots is $-k$.

Hence, $2 + \frac{3}{2} = \frac{7}{2} = -k$.

and $k = -\frac{7}{2}$.

17. $x^2 + 5x + 2k = 0$, where one root is 2.

18. $x^2 + kx + 14 = 0$, where the difference between the roots is 9.

19. $2x^2 - 6kx + 9k = 0$, where one root is 6 times the other.

20. $6x^2 + kx - 7 = 0$, where the quotient of the two roots is $\frac{-21}{2}$.

83. Graph of the quadratic function. To graph the quadratic function $x^2 + x - 2$, compute a table as follows:

$x=$	$-3\frac{1}{2}$	-3	$-2\frac{1}{2}$	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2
$x^2 + x - 2 =$	$6\frac{3}{4}$	4	$1\frac{3}{4}$	0	$-1\frac{1}{4}$	-2	$-2\frac{1}{4}$	-2	$-1\frac{1}{4}$	0	$1\frac{3}{4}$	4

Plotting the points $(-3\frac{1}{2}, 6\frac{3}{4})$, $(-3, 4)$, $\cdots$ from the table and drawing a smooth curve through them we have the curve in Fig. 36. The graph crosses the X-axis at -2 and 1, hence for these values of x the function $x^2 + x - 2$ is zero. In other words, -2 and 1 are the solutions of the equation

$$x^2 + x - 2 = 0.$$

They are represented graphically by the abscissas of the points where the graph crosses the X-axis.

It can be shown, if a is positive and different from zero, that the graph of the function $ax^2 + bx + c$ has the same general characteristics as the curve in Fig. 37. This curve is called a parabola. The real roots of the equation

$$ax^2 + bx + c = 0$$

are given by the abscissas of the points where the curve crosses the X-axis. If the curve has no point in common with the X-axis, then the roots of the equation are imaginary. If the curve touches the X-axis, both roots of the equation are real and equal. These three cases are shown in Fig. 38, where the graphs of $x^2 + x - 2$, $x^2 + x + \frac{1}{4}$, $x^2 + x + 2$ are shown.

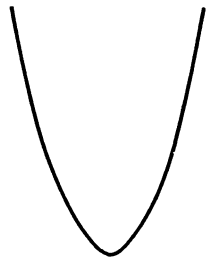

FIG. 37

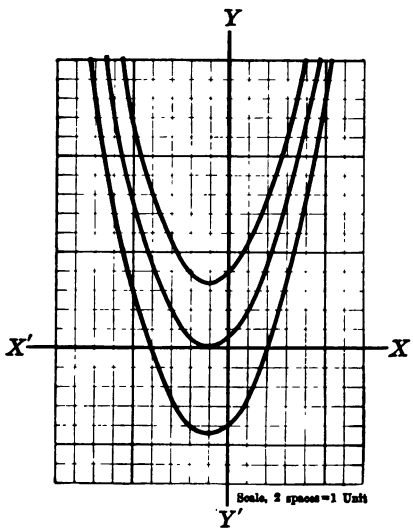

FIG. 38

EXERCISES

Construct the graphs of the functions in the following equations, and by measuring determine the roots if they are real:

1. $x^2 - 4 = 0$.
2. $x^2 - 2x - 3 = 0$.
3. $x^2 - 2x + 5 = 0$.
4. $x^2 - 2x + 1 = 0$.
5. $2x^2 + 3x - 2 = 0$.
6. $2x^2 + 3x + 2 = 0$.
7. $4x^2 - 12x + 9 = 0$.
8. $1 + 2x - x^2 = 0$.
9. $x^2 - 3x = 0$.
10. $3x - x^2 = 0$.

MISCELLANEOUS EXERCISES AND PROBLEMS

Solve:

1. $x^2 - 9x + 14 = 0.$
2. $x^2 + 3x + 11 = 0.$
3. $12x^2 - x - 1 = 0.$
4. $x^2 + 5x - 1 = 0.$
5. $5x^2 + 24x - 5 = 0.$
6. $(x + 1)^2 + 3(x + 1) + 2 = 0.$
7. $3x + \frac{3}{x} = 10.$
8. $\frac{3x - 6}{x + 2} = x - 2.$
9. $\frac{1}{2 + x} = \frac{1}{2} - \frac{1}{x}.$
10. $\frac{1}{a + x} = \frac{1}{a} - \frac{1}{x}.$
11. $\frac{2}{\frac{2}{x} + 3} + \frac{3}{\frac{3}{x} + 2} = 1.$
12. $\frac{n}{\frac{n}{x} + m} + \frac{m}{\frac{m}{x} + n} = 1.$
13. $\frac{x - a}{b} = \frac{2(a - x) - b}{x - a}.$
14. $\frac{x}{a} + \frac{a}{x} = \frac{4}{\frac{x}{a} + \frac{a}{x}}.$
15. Find all of the roots of the equation $x^2 - 9x = 0.$

Solve the following equations:

16. $x^2 - 7.35x + 2.45 = 0.$
17. $2x^2 + .02x - 8.04 = 0.$

What are the values of k if the following equations have equal roots?

18. $x^2 - 2kx + 3k^2 = 5.$
19. $kx^2 + 2kx - 3x + 2 = 0.$
20. Find by the graphical method the approximate values of the roots of the equation $x^2 - 4x - 13 = 0.$

Find the quadratic equations whose roots are:

21. $7, \frac{1}{7}.$
22. $(a - b), (a + 5b).$
23. $n(1 + n), n(1 - n).$

24. For what value of k will the equation

$$\frac{x^2 - 2x}{ax - b} = \frac{k - 1}{k + 1}$$

have roots arithmetically equal but opposite in sign?

25. Find two numbers whose sum is 12 and the sum of whose squares is 74.

26. What numbers differing by 9 have a product 198?

27. Find two consecutive integers whose product is 506.

28. Find two consecutive even integers whose product is 288.

29. Separate 42 into two parts such that the first is the square of the second.

30. Separate the number a into two parts such that the first is the square of the second.

31. One leg of a right triangle is twice the length of the other. The number of feet in the perimeter of the triangle is the same as the number of square feet in the area. Find the sides of the triangle.

32. One leg of a right triangle is n times the length of the other. The measure of the perimeter in feet is the same as the measure of the area in square feet. Find the sides of the triangle.

33. One leg of a right triangle is 5 feet and the hypotenuse is 5 feet longer than one half the other side. What is the area of the triangle?

34. The length of one leg of a right triangle is a and the hypotenuse is a feet longer than one half the other side. What is the area of the triangle?

35. Find the radius of a circle such that the number of feet in its circumference equals the number of square feet in its area.

36. By increasing the radius of a sphere by 1 inch, its volume is increased 40 cubic inches. Find the radius of the original sphere. (See 17, p. 64.)

37. The area of a triangle is 16 square feet and the base is 12 feet more than twice the altitude. Find the base and altitude.

38. The area of a triangle is b square feet and the base is n feet more than twice the altitude. Find the base and altitude.

39. Show that the equation $x^2 - ax - b^2 = 0$ has one positive and one negative root.

40. Around a rectangular flower bed which is 3 yards by 4 yards, there extends a border of turf which is everywhere of equal breadth and whose area is 10 times the area of the bed. How wide is the border?

41. Graph on the same sheet of coördinate paper the function

$$x^2 + x + c$$

where c takes on the values -3, 0, 1, 10. What effect does changing the constant term in a quadratic function have on the graph?

42. Graph on the same sheet the function

$$ax^2 + x + 1$$

where a takes on the values 10, 1, $\frac{1}{2}$, $\frac{1}{10}$. Decreasing the coefficient of x^2 has what effect on the graph?

43. The edges of a cube are each increased in length 1 inch. It is found that the volume is thereby increased 331 cubic inches. What was the length of the edge of the cube?

44. A cistern is filled by two pipes in 40 minutes and the larger pipe can fill it in one hour less time than the smaller one. In how many minutes can the smaller pipe fill the cistern?

45. A rectangular piece of tin is three times as long as it is wide. From each corner a 3-inch square is cut out and the ends turned up so as to make a box whose cubic contents are 312 cubic inches. What were the dimensions of the piece of tin?

46. A man sold some railroad shares for \$4050. A few days later, the price of the shares having dropped \$6 per share, he

used the money to buy 4 more shares than he had sold. How many shares had the man in the first place?

47. The sum of the numerator and denominator of a fraction is 20. If the numerator is decreased by 3 and the denominator increased by 3, the resulting fraction is half a great as the original. What is the fraction?

48. If \$4000 amounts to \$4410 when put at compound interest for two years, interest being compounded annually, what is the rate of interest?

49. A crucible contains 250 pounds of melted lead. A certain quantity of the lead is dipped out and replaced by the same amount of melted tin. From the mixture the same amount is removed and again replaced by the tin. The final mixture contains 16 parts of lead to 9 parts of tin. How much lead was dipped out each time?

50. A and B together can address 180 envelopes in an hour. When each works separately at 180 envelopes, A finishes 27 minutes before B. How many envelopes can each address in an hour?

51. In joining together two steel boiler plates with a single row of rivets, the distance p between the centers of the rivets is given by the formula

$$p = 0.56\frac{d^2}{t} + d,$$

where t is the thickness of the plate and d the diameter of the rivet holes. In a boiler the rivets are to be placed 2 inches apart. If the thickness of the plate is $\frac{1}{4}$ inch, what is the diameter of the rivet holes to the nearest sixteenth of an inch?

52. Work Problem 51 for rivets 1 inch apart.

53. A company of boy scouts engaged to carry a message 84 miles by running in relays. Six of the company were unable to run and it was found that each of the others would have to run one third of a mile more than had been expected. How many boys were in the company?

54. The density of dilute sulphuric acid when 100 grams contain s grams of the pure acid, is given by solving the equation,

$$1.35D^2 + 112D - 111 - s = 0,$$

for D. If $s = 3.5$, find D to two significant figures.

55. The temperature drop below 212°, at which water boils at elevation h above sea level, is obtained by solving

$$t^2 + 517t - h = 0$$

for t. At what temperature does water boil at an elevation of 10,000 feet?

CHAPTER X

SYSTEMS OF EQUATIONS INVOLVING QUADRATICS

84. Introduction. A quadratic equation in two unknowns x and y is an equation that may be written in the form

$$ax^2 + by^2 + cxy + dx + ey + f = 0,$$

where a, b, and c are not all equal to zero. Thus, $x^2 + y^2 = 10$, $y^2 + x = 5$, $xy = 1$, and $xy + 2x + y = 0$ are quadratic equations in two unknowns.

We shall be concerned in this chapter only with quadratic equations of certain special forms. The solution of equations of these special forms will be useful in solving some interesting problems.

85. One equation linear and one quadratic. Any system of equations in two unknowns in which one equation is linear and the other is quadratic, can be solved by a method of substitution.

Example 1. Solve the system

$$x^2 + y^2 = 25, \tag{1}$$

$$x - y = 1. \tag{2}$$

SOLUTION: Solving (2) for y, $y = x - 1$. (3)

Substituting $x - 1$ for y in (1),

$$x^2 + (x - 1)^2 = 25. \tag{4}$$

From (4) $2x^2 - 2x - 24 = 0,$

or $x^2 - x - 12 = 0.$ (5)

Solving (5) by the formula

$$x = \frac{1 \pm \sqrt{1 + 48}}{2}$$

$$= 4 \text{ or } -3.$$

Substituting 4 for x in (2), we obtain $y = 3$.

Substituting -3 for x in (2), we obtain $y = -4$.

This gives $\begin{cases} x = 4 \\ y = 3 \end{cases}$ and $\begin{cases} x = -3 \\ y = -4 \end{cases}$ for the solutions.

Check these solutions by substitution in (1) and (2).

Graphical meaning of the two solutions. We may plot the graphs for each of the equations (1) and (2). The graph of $x - y = 1$ is the straight line shown in Fig. 39, and the graph of $x^2 + y^2 = 25$ is the circle there shown. To draw the graph of (1), the student may give various values to x and calculate the corresponding values for y from $y = \pm\sqrt{25 - x^2}$.

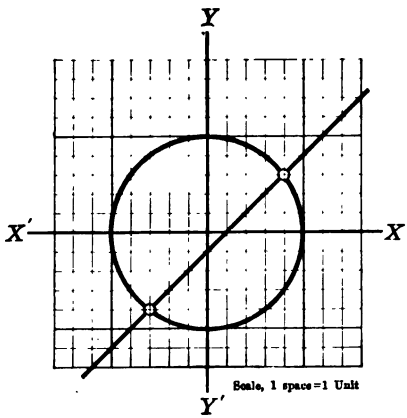

FIG. 39

Any point on the straight line (2) has coördinates that satisfy equation (2). Any point on the circle (1) has coördinates that satisfy equation (1). The points (4, 3) and (−3, −4) lie on both graphs, and satisfy both equations (1) and (2). That is to say, each point of intersection of the graph of (1) with the graph of (2) gives a pair of numbers that is a solution of the system.

Example 2. Solve the system

$$xy = 24, \qquad (1)$$

$$y - 2x + 2 = 0. \qquad (2)$$

SOLUTION: Solving (2) for y in terms of x,

$$y = 2x - 2. \qquad (3)$$

Substituting $2x - 2$ for y in (1),

$$x(2x - 2) = 24, \qquad (4)$$

$$2x^2 - 2x - 24 = 0,$$

$$x^2 - x - 12 = 0,$$

or $$(x + 3)(x - 4) = 0, \qquad (5)$$

or $$x = 4 \text{ or } -3.$$

Substituting 4 for x in (1), we have

$$y = 6.$$

Substituting -3 for x in (1)

$$y = -8.$$

This gives $\begin{cases} x = 4 \\ y = 6 \end{cases}$ and $\begin{cases} x = -3, \\ y = -8 \end{cases}$ for solutions as may be verified by substitution in (1) and (2).

Graphical meaning of the solutions. The graphs for equations (1) and (2) are shown in Fig. 40. The graph of $y - 2x + 2 = 0$ is the straight line,

and that of $xy = 24$ is the curve with two branches as shown. This curve belongs to a class of curves called **hyperbolas.** The points of intersection, P and Q, have coördinates that are the solutions of the given system. All points on the graph of (1) have coördinates that satisfy equation (1). All points on the graph of (2) have coördinates that satisfy equation (2). Therefore, the points of intersection have coördinates that satisfy both equations.

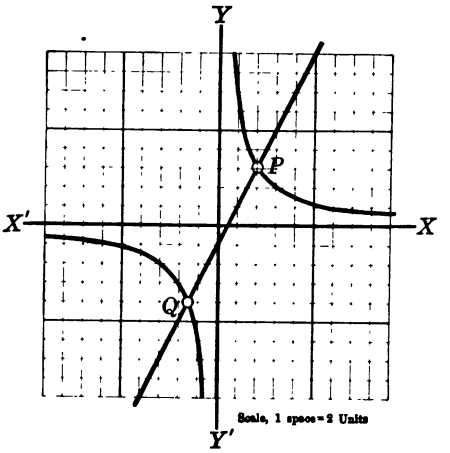

FIG. 40

Example 3. Solve the system

$$x^2 + y^2 = 25, \quad (1)$$
$$x + y = 10, \quad (2)$$

and draw the graph to explain the fact that the solutions are not real.

If the student will carry out the same method as that illustrated in Example 1, he will obtain for solutions

$$x = 5 + \frac{5i}{2}\sqrt{2},$$

and $$x = 5 - \frac{5i}{2}\sqrt{2},$$

$$y = 5 - \frac{5i}{2}\sqrt{2},$$

and $$y = 5 + \frac{5i}{2}\sqrt{2},$$

where $i^2 = -1$.

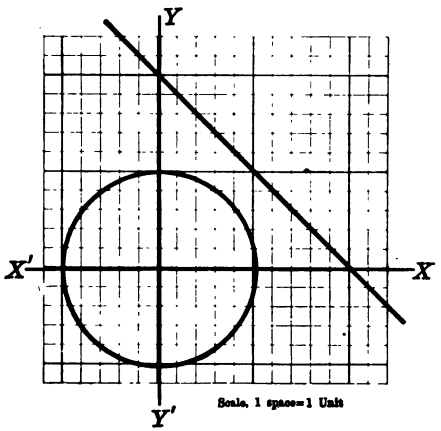

FIG. 41

Check these solutions by substitution in (1) and (2).

The graph of equation (1) is the circle shown in Fig. 41, and the graph of equation (2) is the straight line there shown. It is to be noted that these graphs do not intersect. This fact means that there exists no pair of real numbers that satisfies both equations (1) and (2).

EXERCISES AND PROBLEMS

Solve the following systems, verify each set of roots by substitution in the given equations, and draw the graph for at least Exercises 2, 3, 5, to show the graphic interpretation of the results.

1. $x^2 + y^2 = 13$,
   $x + y = 5$.
2. $x^2 - y^2 = 48$,
   $x - 7y = 0$.
3. $x^2 + y^2 = 13$,
   $2x + 3y = 13$.
4. $5x^2 - 3y^2 + 7 = 0$,
   $2x + y - 7 = 0$.
5. $xy + 54 = 0$,
   $x + y + 3 = 0$.
6. $x + y = 2$,
   $xy + 15 = 0$.
7. $x - y = 1$,
   $x^2 + y^2 = 113$.
8. $4x - 3y + 16 = 0$,
   $\frac{6}{x} + \frac{4}{y} = 1$.
9. $x - y = 1$,
   $xy = a^2 + a$.
10. $x^2 + y^2 = 260$,
    $x - y + 14 = 0$.
11. $xy + 150 = 0$,
    $x - y + 31 = 0$.
12. $2x^2 - 2xy + y^2 = 10$,
    $x + y = 7$.
13. $x - y = 4$,
    $3x^2 + 2xy + 4y^2 = 25$.
14. $x + y + xy = -4$,
    $x - y = 4$.
15. $\frac{1}{x} + \frac{1}{y} = 1$,
    $xy = 4$.
16. $\frac{1}{x} + \frac{1}{y} = 5$,
    $4x + 3y = 3$.

17. The perimeter of a rectangular athletic field is 1124 yards, and the area is 16 acres. What are the dimensions?

18. The difference of the two legs of a right triangle is 7, the hypotenuse is 17. Find the sides of the triangle.

19. Find two consecutive integers the difference of whose squares is 31.

20. The difference of the areas of two squares is 700 square feet and the difference of their perimeters is 40 feet. Find a side of each square.

86. Both equations quadratic. When both equations of a system are quadratic, the problem is often so difficult that the system cannot be solved by methods at present at our disposal. There are, however, some forms of such equations for which we may easily obtain solutions. In Arts. 87 and 88 we shall consider a few such equations.

87. Both equations of the form $ax^2 + by^2 + c = 0$. If, instead of considering x and y as the unknowns, we consider

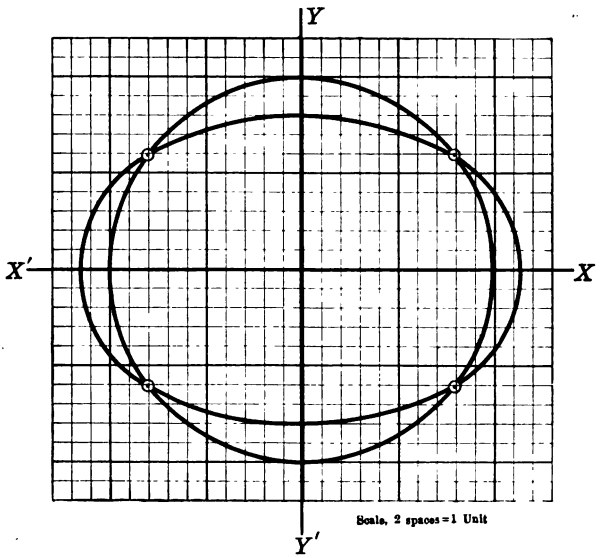

FIG. 42

first x^2 and y^2 as the unknowns, the method of solution is that for linear equations.

Example. Solve $9x^2 + 16y^2 = 288$, (1)

$x^2 + y^2 = 25$. (2)

SOLUTION: Solving for x^2 and y^2, we have

$x^2 = 16$, (3)

$y^2 = 9$. (4)

From (3) and (4), $x = \pm 4, y = \pm 3$.

Forming all possible pairs, we have the four solutions

$$\begin{cases} x = 4 \\ y = 3, \end{cases} \quad \begin{cases} x = 4 \\ y = -3, \end{cases} \quad \begin{cases} x = -4 \\ y = 3, \end{cases} \quad \begin{cases} x = -4 \\ y = -3. \end{cases}$$

The equation (1) has for its locus an oval shaped figure called an **ellipse** (Fig. 42.) The equation (2) has a circle for its locus. The four points of intersection represent graphically the four solutions.

EXERCISES

Solve the following systems of equations and give the graphical representation for 5, 6, 7, and 8.

1. $9x^2 + 25y^2 = 225,$
   $x^2 + y^2 = 25.$
2. $9x^2 + 16y^2 = 160,$
   $x^2 - y^2 = 15.$
3. $4x^2 - 9y^2 = 19,$
   $x^2 + y^2 = 34.$
4. $4x^2 + y^2 = 61,$
   $2x^2 + 3y^2 = 93.$
5. $9x^2 + 16y^2 = 288,$
   $x^2 + y^2 = 9.$
6. $9x^2 + 16y^2 = 288,$
   $x^2 + y^2 = 18.$
7. $9x^2 + 16y^2 = 288,$
   $x^2 + y^2 = 32.$
8. $9x^2 + 16y^2 = 288,$
   $x^2 + y^2 = 64.$

88. Further special methods. Many systems may be solved by special devices in which the aim is to find values for any two of the expressions, $x + y$, $x - y$, and xy, from which the values of x and y may be obtained. Other devices, such as substitution, are suggested in the exercises which follow.

Example 1. Solve the system

$$x^2 + xy = 12, \tag{1}$$
$$y^2 + xy = 4. \tag{2}$$

SOLUTION:

Adding (1) and (2), $x^2 + 2xy + y^2 = 16.$ (3)

Hence, $x + y = \pm 4.$ (4)

Subtracting (2) from (1), $x^2 - y^2 = 8.$ (5)

(5) $\div$ (4) gives, $x - y = +2$ or $-2.$ (6)

From (4) and (6), $x = 3$ or -3; $y = 1$ or $-1.$

By substitution in (1) and (2), we find that only the two pairs

$$\begin{cases} x = 3 \\ y = 1, \end{cases} \text{ and } \begin{cases} x = -3 \\ y = -1 \end{cases} \text{ satisfy (1) and (2).}$$

Example 2. Solve the system

$$x^2 + y^2 = 16, \tag{1}$$

$$y^2 = 6x. \tag{2}$$

SOLUTION: Substituting $6x$ for y^2 in (1), we have

$$x^2 + 6x = 16,$$

or

$$x^2 + 6x - 16 = 0. \tag{3}$$

By formula (Art. 79), $x = \dfrac{-6 \pm \sqrt{36 + 64}}{2}$

$$= 2 \text{ or } -8.$$

If $x = 2$, we have $y = \pm\sqrt{12} = \pm 2\sqrt{3}$.

If $x = -8$, we get $y = \pm\sqrt{-48} = \pm 4i\sqrt{3}$, which are imaginary numbers.

The solutions are $\begin{cases} x = 2 \\ y = 2\sqrt{3}, \end{cases} \begin{cases} x = 2 \\ y = -2\sqrt{3}, \end{cases} \begin{cases} x = -8 \\ y = 4i\sqrt{3}, \end{cases} \begin{cases} x = -8 \\ y = -4i\sqrt{3}. \end{cases}$

Check the results by substitution.

EXERCISES

Solve the following systems:

1. $y^2 = 6x$, $x^2 - y^2 + 8 = 0$.
2. $x^2 + y^2 = 85$, $x^2 - y^2 = 77$.
3. $x^2 + y^2 = 25$, $xy = 12$.
4. $5x^2 - 9y^2 + 121 = 0$, $7y^2 - 3x^2 - 105 = 0$.
5. $x^2 + y^2 = 50$, $xy = -7$.
6. $x^2 + xy = 36$, $xy + y^2 = 45$.
7. $x^2 + xy = 10$, $xy + y^2 = 6$.
8. $p^2 - q^2 = 9$, $4p^2 = 25q$.
9. $s^2 + t^2 = 34$, $st = 15$.
10. $s^2 + st = 45$, $s^2 - st = 5$.

MISCELLANEOUS SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS

Solve:

1. $x^2 + y^2 = 100,$
   $3x + 4y = 50.$

2. $x - y = 5,$
   $xy = -6.$

3. $9x^2 + 25y^2 = 225,$
   $9x^2 - 25y^2 = 225.$

4. $4x^2 = 9y^2,$
   $4x^2 + 9y^2 = 1.$

5. $x^2 + y^2 = 5,$
   $x^2 - xy + y^2 = 3.$

HINT: Subtract second from first.

6. $x + y = 7,$
   $xy - 10 = 0.$

7. $2(x - y)^2 + 2(x + y)^2 = 85,$
   $2(x - y) + 4(x + y) = 25.$

8. $xy - 12 = 0,$
   $x^2 + y^2 - 25 = 0.$

9. $x^2 - y^2 = 3,$
   $x - y = 1.$

HINT: Divide first by second.

10. $4x^2 - y^2 = 16,$
    $2x + y = 8.$

11. $x + y = 2xy,$
    $x + y + x^2 + y^2 = 49.$

12. $x^2 - 2xy - 24y^2 = 32,$
    $x - 6y = 2.$

13. $x^2 + 4y^2 = 24,$
    $xy = 6.$

14. $x - y = 0,$
    $x + y^2 = 20.$

15. $2x^2 - 3xy - y^2 = 1,$
    $5x + y = 3.$

16. $x - xy = 5,$
    $2y + xy = 6.$

17. $x^2 + y^2 + 3x + 3y = 28,$
    $xy - 6 = 0.$

HINT: Multiply second by 2, add to first and solve for $x + y$.

18. $x^2 + y^2 + x + y = 2,$
    $xy - 2 = 0.$

19. $xy = 2(x + y),$
    $x^2 + 2xy + y^2 = 1.$

20. $2x^2 + 5y^2 = 125,$
    $5x + 2y = 10.$

21. $2x^2 - xy = 10,$
    $3x^2 - y^2 = 11.$

HINT: Multiply first by 11, second by 10, and subtract to eliminate the constant terms. Factor the resulting expression into

$$(x + 2y)\ (8x - 5y) = 0.$$

22. $x^2 + xy + 2y^2 = 44,$
    $2x^2 - xy + y^2 = 16.$

23. $x^2 + y^2 = 16,$
    $\frac{x^2}{9} + \frac{y^2}{16} = 1.$

24. $x^2 + y^2 = 25,$
    $3x - 2y = 6.$

25. $x^2 + 4y^2 = 17,$
$xy = 2.$

26. $x^2 + y^2 = 9,$
$y^2 = 4 - x^2.$

27. $x^2 - xy = 54,$
$xy - y^2 = 18.$

28. $3x^2 + 3xy + 2y^2 = 8,$
$x^2 - xy - 4y^2 = 2.$

HINT: Let $y = vx$. Substitute in both equations and solve for v and x.

29. $x^2 + 3xy - 2y^2 = 2,$
$2x^2 - 5xy + 6y^2 = 3.$

30. $x^2 + xy - 6y^2 = 0,$
$x^2 + 4x + 3y = 69.$

HINT: Solve the first equation for x in terms of y, and substitute in second.

31. $6x^2 + 5xy - 6y^2 = 0,$
$2x^2 - y^2 + 5x = 9.$

32. $x^2 + 3xy + y^2 = 41,$
$x^2 + y^2 + x + y = 32.$

33. $x^2 + 3xy + 2y^2 = 3,$
$2x^2 + y^2 = 6.$

34. $(x + y)^2 - 3(x + y) = 10,$
$x^2 - y^2 = 5.$

35. $x^2 + xy = 6,$
$xy - y^2 = 1.$

36. $x^2 + y^2 + x - y - 12 = 0,$
$x^2 - xy = 0.$

37. $x^2 + 2xy + y^2 + 2x + 2y + 1 = 0,$
$x^2 - 2xy + y^2 - 2x + 2y + 1 = 0.$

38. The sum of two numbers is 12 and their product is 35. Find the numbers.

39. The sum of the squares of two numbers is 100, and their difference is 2. Find the numbers.

40. Find two numbers whose product is 150 and whose quotient is $\frac{3}{2}$.

41. The area of a rectangle is 120 square yards and the diagonal is 17 yards. Find the sides.

42. A rope 48 feet long exactly surrounds an enclosure in the form of a right triangle of hypotenuse 20 feet. Find the other sides of the enclosure.

43. A loop of twine 34 inches long is to be put around four pegs set in the form of a rectangle of area 60 square inches. Find the sides of the rectangle.

44. A farmer raised broom corn and pressed 6120 pounds into bales. If he had made each bale 20 pounds heavier, he would have had one bale less. How many bales did he press and what was the weight of each?

45. After a mowing machine had made the circuit of a 10-acre rectangular field 33 times, cutting a swath 5 feet wide each time, $2\frac{1}{2}$ acres of grass were still standing. Find the dimensions of the field.

46. A farmer bought 5 cows and 30 sheep for \$525. He bought one more sheep for \$25 than he bought cows for \$300. Find the price of each per head.

47. A field contains 9 acres. If its length were decreased by 20 rods and its width by 4 rods, its area would be less by 4 acres. Find the length and width.

48. If the length of a diagonal of a rectangular field of 30 acres is 100 rods, how many rods of fence will be required to enclose the field?

49. It took a number of men as many days to pave a street as there were men, but had there been five more workmen employed, the work would have been done 4 days sooner. How many men were employed?

50. The sum of the squares of two consecutive integers is 1301. Find the numbers.

51. A rectangular piece of tin containing 400 square inches is made into an open box, containing 384 cubic inches, by cutting out a 6-inch square from each corner of the tin and then folding up the sides. Find the dimensions of the original piece of tin.

52. If the product of two numbers is increased by their sum, the result is 79. If their product is diminished by their sum, the result is 47. Find the numbers.

53. For two numbers a and b it is found that the expressions

$$\frac{3ab}{a+b},\ \frac{a}{b}(a-b),\ a^2-b^2,$$

give the same number. What is this number?

54. The area of a field is 367,500 square feet. The length of the diagonal is 875 feet. What are the dimensions?

55. The difference of the squares of two numbers is divided by the smaller number. The quotient is 4, and the remainder is 4. If the difference of the squares of the numbers is divided by the greater number the quotient is 3 and the remainder 3. What are the numbers?

56. A certain floor having an area of 96 square feet can be covered with 432 rectangular tiles. If the workman use a tile one inch longer but one inch less in width, it takes 512 tiles to cover the floor. What are the dimensions of the two sizes of tiles?

57. A club of boys bought a motor boat for \$192. Four boys failed to pay their share as agreed, so each of the others was compelled to pay \$4 more than he had promised. How many boys were in the club?

58. A father divided \$1000 between his two sons and kept it for them at simple interest until called for. At the end of $2\frac{1}{2}$ years, one son called for all the money due him and received \$632.50. At the end of 3 years the other son received \$531 as his share. How was the money originally divided and what interest did the father pay?

59. A few days after the outbreak of the European war the cost of 100 pounds of sugar in the United States was \$3.10 more than it had been just before the outbreak. For \$330 a grocer received 3100 pounds less sugar after the outbreak than he would have received before for the same amount. What was the price before and after the outbreak of the war?

CHAPTER XI

EXPONENTS, RADICALS, AND ROOTS

89. Positive integral exponents. The expression a^n, when n is a positive integer, means the product $a \cdot a \cdot a \ldots$ to n factors.

In operating with positive integral powers we make use of the following laws:

I. $a^m \cdot a^n = a^{m+n}$.

Example. $x^2 \cdot x^3 = x^5$.

II. $a^m \div a^n = a^{m-n}$, where $m > n$.

Example. $4^7 \div 4^3 = 4^4$.

III. $(a^m)^n = a^{mn}$.

Example. $(b^4)^6 = b^{24}$.

IV. $(a^p b^q c^r \cdot \cdot \cdot)^n = a^{pn} b^{qn} c^{rn} \cdot \cdot \cdot$.

Example. $(5^2 a^3 c^7)^4 = 5^8 a^{12} c^{28}$.

V. $\left(\frac{a^m}{a^n}\right)^p = \frac{a^{mp}}{a^{np}}$.

Example. $\left(\frac{a^7}{b^4}\right)^5 = \frac{a^{35}}{b^{20}}$.

EXERCISES

Perform the indicated operations and simplify the results when possible. Fractions should be reduced to lowest terms.

1. $a^2 \cdot a^5$.
2. $a^4 b^2 \cdot a^7 b^3$.
3. $\frac{a^7}{a^4}$.
4. $\frac{a^7}{a^{12}}$.
5. $h^n \cdot h^3$.
6. $\frac{h^{n+7}}{h^4}$.

7. $\dfrac{6m^4n^6p^p}{2mn^2p^3}$.

8. $\dfrac{12a^4b^2c}{8ab^3c^2}$.

9. $(3a^3b^2c)^2$.

10. $(-3m^2n^3)^3$.

11. $\left(\dfrac{4ab}{m^2n}\right)^2$.

12. $\left(\dfrac{-2m}{5xy}\right)^2$.

13. $a^7 \cdot a^{x+4}$.

14. $(-3)^2 \cdot (-3)^3$.

15. $\dfrac{a^{x+4}b^6}{a^2b^6}$.

16. $\dfrac{a^{3n}x^{n+1}}{a^nx^3}$.

17. $(2a^{2n})^3$.

18. $(a^2b^3)^n$.

19. $(r^ns^{2n})^4$.

20. $(r^ns^{2n})^n$.

21. $\dfrac{a^{4n}b^m}{a^nb^3}$.

22. $\dfrac{a^{x+2}}{a^{x-2}}$.

23. $\dfrac{(a+b)^3}{a^3+b^3}$.

24. $\left(\dfrac{6a}{b}\right)^2 \cdot \left(\dfrac{b}{2a}\right)^3 \cdot \left(\dfrac{b}{3a}\right)^2$.

25. $\left(\dfrac{a^2}{x^3}\right)^n \cdot \left(\dfrac{c^2}{y^3}\right)^n \cdot \left(\dfrac{x^3y^3}{ac^2}\right)^n$.

26. $\dfrac{(x^4-y^4)^2}{(x^2+y^2)^3} \cdot \dfrac{x^8-y^8}{(x^4+y^4)^2}$.

27. $\dfrac{6a^{2x-3y}\,b^{3y-5}}{(2a^{x-2y}\,b^{y-9})^3}$.

28. $\dfrac{(a-1)^{3n}}{(1-a)^{2n}}$.

29. $\dfrac{(6a^2x^3y)^2}{(4ax^2y^2)^3}$.

30. $(1\frac{1}{3})^3 \cdot (1\frac{1}{2})^3 \cdot (\frac{1}{2})^3$.

31. Find the value of $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3} + \dfrac{1}{x^4}$ for $x = 10$.

32. Find the value of $3x + 5 + \dfrac{4}{x} + \dfrac{9}{x^2} + \dfrac{2}{x^3}$ for $x = 10$.

33. Find the value of $2x^3 + x + \dfrac{1}{x^2} + \dfrac{8}{x^4}$ for $x = 10$.

34. Find the value of each of the following sums for $x = \frac{1}{2}$:

(*a*) $1 + x$.
(*b*) $1 + x + x^2$.
(*c*) $1 + x + x^2 + x^3$.
(*d*) $1 + x + x^2 + x^3 + x^4$.

35. Find the sum of 2, of 3, of 4, and of 5 terms of

$$1 - \frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} - \ldots, \text{ for } x = 10.$$

36. Find the value of $5t^3 + 6t^2 + 9t + 3$ for $t = 10$. Also of $2t + 5 + \frac{1}{t} + \frac{3}{t^2} + \frac{5}{t^3}$.

37. Write in terms of powers of $t = 10$, the numbers 8643, 48609, and 100101.

38. Expressed in terms of powers of $t = 10$,

$$35.762 = 3t + 5 + \frac{7}{t} + \frac{6}{t^2} + \frac{2}{t^3}.$$

Write in terms of powers of $t = 10$, the numbers 3.1416, 1.4142, and .06041.

90. Extension of the meaning of an exponent. Thus far we have dealt entirely with positive integral exponents. No meaning has thus far been given to a^n if n is zero, fractional, or negative. It is found useful to give a meaning to such expressions as a^0, $8^{\frac{1}{3}}$, and 2^{-3}. It is convenient to define them so that the laws for positive integral exponents hold for these new kinds of exponents. Such definitions will now be given.

91. Meaning of a^0. In order that Law I, Art. 89, may hold for an exponent zero, it is necessary that

$$a^0 \cdot a^m = a^{0+m} = a^m,$$

or

$$a^0 = 1, \text{ if } a \neq 0.$$

That is, for all values of a, except 0,

$$a^0 = 1.$$

92. Meaning of $a^{\frac{1}{3}}$. Assuming that Law I, Art. 89, is to hold for fractional exponents,

$$a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a,$$

and $a^{\frac{1}{3}}$ is one of the three equal factors whose product is a. But this is what has been defined as the cube root of a. Hence we say that

$$a^{\frac{1}{3}} = \sqrt[3]{a}.$$

In general, if q is a positive integer,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{ to } q \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots \text{ to } q \text{ terms}}$$

$$= a,$$

and $a^{\frac{1}{q}}$ is one of the q equal factors of a, or a qth root of a.

That is, $$a^{\frac{1}{q}} = \sqrt[q]{a}$$

Examples. $$8^{\frac{1}{3}} = \sqrt[3]{8} = 2,$$

$$50^{\frac{1}{2}} = \sqrt{50} = 5\sqrt{2}.$$

93. Meaning of $a^{\frac{p}{q}}$. Let p and q be positive integers. Then applying Law I, Art. 89, we have

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{ to } p \text{ factors} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots \text{ to } p \text{ terms}} = a^{\frac{p}{q}},$$

and $a^{\frac{p}{q}}$ means the pth power of the qth root of a. That is,

$$a^{\frac{p}{q}} = (\sqrt[q]{a})^p = \sqrt[q]{a^p}.$$

Examples. $$27^{\frac{2}{3}} = (\sqrt[3]{27})^2 = 9,$$

$$9^{\frac{5}{2}} = (\sqrt{9})^5 = 243.$$

We have now found meanings for zero and fractional exponents.

94. Meaning of a^m when m is negative. Let $m = -n$, where n is a positive number. Applying Law I, Art. 89, we have

$$a^n \cdot a^{-n} = a^{n-n} = a^0 = 1.$$

Hence, $$a^{-n} = \frac{1}{a^n}, \text{ if } a \neq 0,$$

and $$a^m = \frac{1}{a^{-m}}.$$

Thus, a^m and a^{-m} are reciprocals of each other.

Historical note on exponents. It appears that Nicole Oresme, a bishop of Normandy, in the fourteenth century conceived the idea of fractional exponents, and gave some rules for operating with them. But his notation was very different from ours. To Simon Stevin of Bruges in Belgium (1548–1620) belongs the honor of inventing our present method of de-

Therefore in a fraction, any factor of the numerator may be changed to the denominator, or any factor of the denominator to the numerator, if the sign of the exponent of the factor is changed.

Example. $\dfrac{2^2a^{-2}b}{5^{-2}x^3} = \dfrac{2^2 \cdot 5^2 b}{a^2x^3}$.

We have now found meanings for zero, fractional, and negative exponents.

EXERCISES

Give in simplest form the values of the following:

1. $5^0 \cdot 36^{\frac{1}{2}}$.
2. $5a^0$.
3. $9 \cdot 30^{-2}$.
4. $\dfrac{1}{4^{-2}}$.
5. $\dfrac{5}{2^{-1}}$.
6. $\left(\dfrac{1}{5}\right)^{-2}$.
7. $(-.5)^3$.
8. $(x+2)^0$.
9. $(-.1)^{-1}$.
10. $16^{\frac{1}{2}}$.
11. $8^{\frac{2}{3}}$.
12. $25^{\frac{1}{2}} \cdot 4^{-\frac{1}{2}}$.
13. 1^n.
14. 1^{-n}.
15. $\left(\dfrac{1}{25}\right)^{-\frac{1}{2}}$.
16. $(5\frac{1}{16})^{\frac{1}{4}}$.
17. $(\frac{3}{4})^0 \cdot 32^2$.
18. $(.125)^{\frac{1}{3}}$.
19. $64^{\frac{1}{3}}$.
20. $8^{-1}(\frac{2}{3})^0 \cdot 36^{\frac{1}{2}}$.
21. $\dfrac{1}{2^{-1}+3^{-1}}$.
22. $\dfrac{2^{-2}-5^{-2}}{10^{-3}}$.
23. $(3\frac{4}{17})^0 + 8^{-\frac{1}{3}} - (\frac{1}{5})^{-1}$.
24. $2 \cdot 3^{-1} - 6^{-1}$.
25. $8 \cdot 27^{\frac{2}{3}} - 27 \cdot 8^{\frac{2}{3}} + 8^{-\frac{2}{3}} \cdot 27^{-\frac{2}{3}} - 8^{\frac{2}{3}} \cdot 27^{\frac{2}{3}}$.
26. $2 \cdot 5^{-1} - 5^{-2} + (\frac{1}{5})^2 - (\frac{5}{2})^{-1}$.

---

noting powers. It remained, however, for John Wallis (1616–1703), an Englishman, to show the advantages of fractional and negative exponents to such an extent as to assure for them a permanent place in algebra.

27. $\dfrac{3 \cdot 5^{-1} - 5 \cdot 3^{-1}}{3^{-1} \cdot 5^{-1}}$.

28. $(1^{-1} + 2^{-2} + 3^{-3})^0$.

29. $3 \cdot 32^{\frac{2}{5}} - 3 \cdot 32^{\frac{1}{5}}$.

30. $\left(\dfrac{1^{-1} + (-1)^{12}}{14^0 - 0^{14}}\right)$.

Change the following into identical expressions without zero or negative exponents; and simplify when possible:

31. a^0x^0.

32. $(x^2 + y^2)^0$.

33. $5a^2x^{-2}y^{-3}$.

SOLUTION: By Art. 94,
$5a^2x^{-2}y^{-3} = 5a^2 \cdot \dfrac{1}{x^2} \cdot \dfrac{1}{y^3} = \dfrac{5a^2}{x^2y^3}$.

34. $\dfrac{a^0}{b^{-4}}$.

35. $3^{-2} - 3^{-1}$.

36. $\dfrac{3^{-2}}{3^{-1}}$.

37. $\dfrac{m^2n^{-4}}{m^{-2}n^{-3}}$.

SOLUTION: By Art. 94,
$\dfrac{m^2n^{-4}}{m^{-2}n^{-3}} = \dfrac{m^4n^3}{n^4} = \dfrac{m^4}{n}$.

38. $\dfrac{12ab^{-3}}{3a^{-3}b}$.

39. $3x^n - \dfrac{3}{x^{-n}}$.

40. $(-x)^{-2} \cdot (-y)^{-3}$.

41. $x^{-\frac{1}{2}}y^{-\frac{1}{3}}$.

42. $\dfrac{a^3b^2c^{-5}}{a^{-1}bc^2}$.

43. $\dfrac{1}{5a^{-3}b^{-5}}$.

44. $\dfrac{x^{-2} + y^{-2}}{x^{-2}y^{-2}}$.

45. $\dfrac{a^4}{a^{-4} + a^{-5}}$.

46. $.1^{-3} \cdot 10^3$.

47. $(a - b)^{-2}\ (a^2 - b^2)$.

48. $(1 - x)^{-2}\ (x - 1)$.

49. $4^{-1}(x + y + z)^0$.

50. $\dfrac{0^3 \cdot a^{-\frac{5}{2}}m^{\frac{7}{2}}}{a^2bm^{-3}}$.

51. $\dfrac{a^{-\frac{1}{2}}b^2c}{a^{\frac{1}{2}}b^{-\frac{1}{2}}c^{\frac{1}{2}}}$.

52. $(x^2 + y)^{-2}\ (x^2 + y)^3$.

53. $a^{-1} + 2a^{-2} + 3a^{-3}$.

54. $\dfrac{1}{a^{-1} + 2a^{-2} + 3a^{-3}}$.

55. $a^{-1} + (2a)^{-2} + (3a)^{-3}$.

56. $\dfrac{1}{x^{-2}} - \dfrac{1}{y^{-2}}$.

57. $\dfrac{1}{x^{-2} - y^{-2}}$.

58. $2a^x - \dfrac{1}{2a^{-x}}$.

59. $\left(\dfrac{1}{3}\right)^{-2} - \left(\dfrac{1}{2}\right)^{-3}$.

60. $8^0 + 8^{-1} + 8^{\frac{1}{3}} - 8^{-\frac{1}{3}}$.

Change the following into identical expressions without fractional or negative exponents:

61. $x^{\frac{1}{2}}$.

62. $a^{\frac{1}{3}}$.

63. $8^{\frac{1}{2}} \cdot 5^{\frac{1}{3}}$.

64. $m^{-\frac{1}{2}}$.

65. $z^{\frac{1}{q}}$.

66. $y^{\frac{n}{m}}$.

67. $\dfrac{3a^{-\frac{1}{2}}}{b^{\frac{1}{3}}}$.

68. $\dfrac{c^2}{5d^{-\frac{1}{3}}}$.

69. $e^{-\frac{m}{n}}$.

70. $x^{\cdot 2}$.

71. $(m+n)^{\frac{1}{2}}$.

72. $m^{\frac{1}{2}} + n^{\frac{1}{2}}$.

73. $x^{\frac{1}{2}}y^{\frac{1}{2}}$.

74. $(8x^3y^6)^{\frac{1}{3}}$.

75. $3m^{\frac{1}{2}}n^{\frac{1}{3}}$.

76. $12a^{-\frac{1}{2}}$.

77. $a^2b^{-\frac{1}{2}}c^{\frac{1}{3}}$.

78. $(r^2+4)^{-\frac{1}{2}}$.

79. $(k^{-1} + l^{-\frac{1}{2}})^{\frac{1}{n}}$.

80. $7x^{-7}y^{\frac{1}{q}}$.

81. $[(a^{\frac{1}{2}})^{\frac{1}{3}}]^{-1}$.

82. $(n^2 - s^2)^{\cdot 5}$.

83. $\dfrac{(3m)^{-\frac{1}{2}}}{2^{-1}}$.

84. $\dfrac{1}{(1+x)^{-1}}$.

85. $\dfrac{1}{1+x^{-1}}$.

86. $y^{-1} + y^{-2} + y^{-3}$.

87. $y^{\frac{1}{2}} + y^{\frac{1}{3}} + y^{\frac{1}{4}}$.

88. $x^{2\cdot 1} + x^{-2\cdot 1}$.

89. $\left(\dfrac{8a^6}{27b^{-9}}\right)^{\frac{1}{3}}$.

90. $\left(\dfrac{2}{3}\right)^{-2}\left(\dfrac{16}{225}\right)^{-\frac{1}{2}}$

Change the following into identical expressions without radical signs or negative exponents:

91. $\sqrt{a}$.

92. $\sqrt[4]{a^3}$.

93. $\sqrt{a^2b^6c^4}$.

94. $\sqrt{a^3b^5c^4}$.

SOLUTION: $\sqrt{a^3b^5c^4} = (a^3b^5c^4)^{\frac{1}{2}}$, by definition of a fractional exponent,

$= a^{\frac{3}{2}}b^{\frac{5}{2}}c^2$, by Law IV, Art. 89.

95. $\sqrt[3]{a} \cdot \sqrt{b^6}$.

96. $\sqrt[n]{a^n b^{2n} c^{3n}}$.

97. $\sqrt[3]{a^6b^2c}$.

98. $\sqrt[4]{a+b}$.

99. $\sqrt[4]{(a+b)^8}$.

100. $\sqrt[3]{a^6+b^6}$.

101. $\sqrt[5]{\dfrac{x^{10}y^5}{m^{15}n^{30}}}$.

102. $\sqrt[3]{\dfrac{ab^2c^3}{x^5yz^4}}$.

103. $\sqrt{a^{\frac{1}{2}}b} \cdot \sqrt[3]{x^2}$.

104. $\sqrt[3]{a^{-6}b^{-2}}$.

105. $\sqrt{a^{-2q}b^{2q}}$.

106. $\sqrt[3]{\sqrt{a^5}}$.

107. $\sqrt[4]{16a^{-8}c^{-3}}$.

108. $\sqrt[3]{8^{-3}\sqrt{r^{-4}s^5}}$.

109. $\sqrt{36+64}$.

110. $\sqrt[3]{9} \cdot \sqrt[5]{5}$.

Perform the indicated operations. By the use of negative exponents if necessary, leave the results in forms without denominators.

111. $a^6 \cdot a^{-3}$.

112. $a^4 \cdot a^3 \cdot a^{-7}$.

113. $x^2y^{-1} \cdot x^{-3}y^2$.

114. $\dfrac{m^3}{m^{-3}}$.

115. $y^n \cdot y^{2-n}$.

116. $a^{-7} \div a^{-13}$.

117. $k^{\frac{1}{2}} \cdot k^{\frac{3}{2}}$.

118. $r^{-2} \cdot r^{\frac{1}{2}}$.

119. $(2a^0b^{-3}) \div (a^{-2}b^2)$.

120. $(a^2)^3$.

121. $(m^{-3})^4$.

122. $(x^{-n})^{-2}$.

123. $\left(\dfrac{r^2}{s}\right)^{\frac{1}{2}}$.

124. $\left(\dfrac{a^2}{b^{-3}}\right)^{\frac{1}{2}}$.

125. $(-a^5)^{-2}$.

126. $(-m^5)^{-3}$.

127. $\left(\dfrac{a^{-1}b^{-3}}{a^2b^2}\right)^2$.

128. $m^x \div m^{-x}$.

129. $r^{t+1} \cdot r^{2t}$.

130. $\left(\dfrac{r^{t+1}}{s^{-3}}\right)^2$.

131. $(r^{t+1})^{t+1}$.

132. $(a^{\frac{1}{2}} - b^{-\frac{1}{2}})^2$.

133. $(a^{\frac{1}{2}} - b^{-\frac{1}{2}})^0$.

134. $(2^{1-n} \cdot 2^{n-1})^{\frac{1}{2}}$.

135. $(x^{\frac{1}{2}})^{\frac{1}{2}}$.

136. $(x^{-1} + x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})(x^{-1} - x^{-\frac{1}{2}}y^{-\frac{1}{2}} + y^{-1})$.

137. $(m^{-p} + n^{-q})(m^{-2p} - m^{-p}\,n^{-q} + n^{-2q})$.

138. $(a^{-\frac{n}{m}} + b^{\frac{p}{q}})^3$.

139. $(ax^{-2} + bx^{-1} + c)^2$.

140. $\left(\dfrac{a}{b} + \dfrac{c}{d}\right)^2$.

95. Rational and irrational numbers. A rational number is defined (Art. 67) as one that can be expressed as the quotient

of two integers. An irrational number is a real number that cannot be thus expressed.

Thus, 16, $\frac{1}{2}$, and $\frac{18}{25}$ are rational numbers; $\sqrt{2}$,* $\sqrt[3]{2}$, $\sqrt{3}$, $\sqrt{5}$, $1+\sqrt{5}$, and $9^{\frac{1}{3}}$ are irrational numbers.

Any irrational number can be enclosed between two rational numbers that differ from one another by as small a number as we please.

Thus, we may write,

$$1<\sqrt{2}<2.$$

$$1.4<\sqrt{2}<1.5.$$

$$1.41<\sqrt{2}<1.42.$$

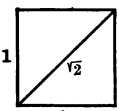

FIG. 43

Either of the two sequences of numbers in the two outer columns determines $\sqrt{2}$ in the same way that .3, .33, .333, . . . , determine $\frac{1}{3}$.

As a geometrical illustration of an irrational number we may take the diagonal of a square whose side is 1.

96. Radicals. An indicated root of a number is called a **radical.**

Thus, $\sqrt{3}$, $\sqrt[3]{64}$, $\sqrt{\sqrt[4]{7}}$, and $\sqrt[n]{a+b}$ are radicals.

The **radicand** is the number of which the root is to be taken.

* To show that $\sqrt{2}$ cannot be expressed as the quotient of two integers, suppose it is possible that

$$\sqrt{2}=\frac{m}{n},$$

where $\frac{m}{n}$ is a rational fraction in its lowest terms. At least one of the numbers m or n is odd. Clearing of fractions and squaring both sides, we get

$$2n^2=m^2.$$

From this equation, we see that m^2 is an even number. Hence m is an even number. If m is even, m^2 contains the factor 4. Hence n^2 is an even number, and n is itself even. This is contrary to hypothesis that $\frac{m}{n}$ is a fraction in its lowest terms.

This proof is found in Euclid (about 300 B.C.), and is supposed to be due to a much earlier mathematician than Euclid.

The number that tells what root is to be taken is called the **index** of the root.

Thus in $\sqrt{3}$, $6\sqrt[4]{19}$, and $8\sqrt[3]{x+y}$, the radicands are 3, 19, and $x+y$, and the indices are 2, 4, and 3, respectively.

The **order** of a radical is given by the index of the root.

Thus $\sqrt{5}$ is of the second order, $\sqrt[3]{85}$ is of the third order, and so on.

97. Surds. A **surd** is an irrational number that is the root of a rational number.

Thus $\sqrt{2}$, $\sqrt[5]{19}$, $\sqrt[3]{45}$ are surds. But $\sqrt[3]{8}$ is not a surd, since $\sqrt[3]{8} = 2$, a rational number; and $\sqrt{2+\sqrt{3}}$ is not a surd because $2+\sqrt{3}$ is not a rational number.

A surd of the second order is called a **quadratic surd.**

EXERCISES

1. Give an example (*a*) of a rational number; (*b*) an irrational number; (*c*) a radical; (*d*) a surd; (*e*) an index; (*f*) a radicand.

2. Give an example of a radical that is not a surd.

3. Name the order of each of the following: $\sqrt{8}$, $\sqrt[3]{a^2}$, $\sqrt[5]{x+y}$.

4. Tell which of the following are rational: 4, $\frac{9}{3}$, $\sqrt{49}$, $\sqrt[3]{9}$, .3, .333, $1+\sqrt{5}$. Tell which are irrational.

5. Tell which of the following are surds: $\sqrt{9}$, $\sqrt[3]{9}$, $\sqrt{5}$, $1+\sqrt{5}$.

6. Give an example of a surd of the second order; of the third order; of the fifth order.

98. Introducing a factor under the radical sign. The coefficient of a radical may be introduced under the radical

sign as a factor of the radicand by raising such coefficient to a power equal to the index of the root.

Example 1. $7\sqrt{2} = \sqrt{7^2} \cdot \sqrt{2} = \sqrt{7^2 \cdot 2} = \sqrt{98}.$

Example 2. $2a^2b\sqrt[3]{abx} = \sqrt[3]{8a^6b^3}\sqrt[3]{abx} = \sqrt[3]{8a^7b^4x}.$

EXERCISES

In the following introduce the coefficients under the radical sign:

1. $5\sqrt{3}.$
2. $2\sqrt[3]{2}.$
3. $6\sqrt{7}.$
4. $3\sqrt[3]{5}.$
5. $10\sqrt[4]{.01}.$
6. $a\sqrt{b}.$
7. $2a^2y\sqrt[3]{x}.$
8. $-5mn\sqrt[3]{3m^2}.$
9. $\frac{1}{2}\sqrt{6}.$
10. $\frac{3}{5}\sqrt[3]{25am}.$
11. $2\sqrt{a+b}.$
12. $(1+x)\sqrt{1+x}.$
13. $(a+b)\sqrt[3]{3}.$
14. $(m-n)\sqrt{m+n}.$
15. $\frac{a}{b}\sqrt[3]{a+b}.$

99. Simplification of radicals. For purposes of computation it is convenient to be able to change the form of radicals.

In the case of square roots, we have, that

$$\sqrt{ab} = (ab)^{\frac{1}{2}} = a^{\frac{1}{2}}\, b^{\frac{1}{2}} = \sqrt{a}\ \sqrt{b}.$$

By Law IV, Art. 89, this principle can be extended. By this law,

$$(a^n b^m)^{\frac{1}{k}} = a^{\frac{n}{k}} b^{\frac{m}{k}},$$

that is,

$$\sqrt[k]{a^n b^m} = \sqrt[k]{a^n}\sqrt[k]{b^m}.$$

This principle is useful in simplifying radicals.

Thus,

$$\sqrt[3]{a^6x^2} = \sqrt[3]{a^6}\ \sqrt[3]{x^2} = a^2\sqrt[3]{x^2}.$$

100. Simplest form of a radical. A radical is said to be in its simplest form if the radicand —

(1) Is integral.

(2) Contains no rational factor raised to a power whose exponent equals the index of the radical.

(3) Is not a power whose exponent has a common factor with the index of the radical.

Thus, $\sqrt{\frac{1}{2}}$, $\sqrt[3]{a^2b^5}$, and $\sqrt[6]{4a^2b^4}$ are not in their simplest forms, because they may be written as $\frac{1}{2}\sqrt{2}$, $b\sqrt[3]{a^2b^2}$, and $\sqrt[3]{2ab^2}$ respectively.

EXERCISES

1. Tell which of the following are not in their simplest forms and give reasons:

$$\sqrt[3]{\frac{5}{6}}, \sqrt{50}, \sqrt[3]{abc}, \sqrt[5]{\frac{a^2b^7}{c^3}}, \sqrt[4]{16m^2n^4}, \sqrt{\frac{24}{17}}, \sqrt[3]{a^2b^3 - a^4b^5}.$$

Reduce to simplest form:

2. $\sqrt{75}$.

SOLUTION: $\sqrt{75} = \sqrt{25 \cdot 5} = \sqrt{25} \cdot \sqrt{5} = 5\sqrt{5}$.

3. $\sqrt{18}$.

4. $\sqrt[3]{81}$.

5. $\sqrt{98}$.

6. $3\sqrt{72}$.

7. $7\sqrt{147}$.

8. $\sqrt[4]{81}$.

9. $5\sqrt[3]{40}$.

10. $\sqrt[3]{\frac{2}{3}}$.

SOLUTION:

$$\sqrt[3]{\frac{2}{3}} = \sqrt[3]{\frac{18}{27}} = \sqrt[3]{\frac{1}{27} \cdot 18} = \frac{1}{3}\sqrt[3]{18}.$$

11. $\sqrt{\frac{1}{2}}$.

12. $\sqrt{\frac{5}{6}}$.

13. $2\sqrt{\frac{4}{7}}$.

14. $5\sqrt[3]{\frac{1}{4}}$.

15. $8\sqrt[4]{\frac{8}{9}}$.

16. $\sqrt{a^2b}$.

17. $\sqrt{4a^2b^3c}$.

18. $\sqrt[3]{16x^5y^3}$.

19. $3\sqrt[3]{24x^6y^5z^4}$.

20. $\sqrt{a^2x^2 + a^3x^4}$.

21. $\sqrt{m^6 + m^3n}$.

22. $\sqrt[3]{(a+b)^4}$.

23. $[(x^2 - y^2)^3]^{\frac{1}{3}}$.

24. $(m^2n^3 + 6m^3n^4)^{\frac{1}{2}}$.

25. $\left(\frac{a^2}{b}\right)^{\frac{1}{2}}$.

26. $\left(\frac{x^3y^4}{ab^4}\right)^{\frac{1}{3}}$.

27. $\left(\frac{2x}{5y}\right)^{\frac{1}{2}}$.

28. $\left(\frac{27n}{5m^2}\right)^{\frac{1}{3}}$.

29. $\left(1 + \frac{a^2}{b^2}\right)^{\frac{1}{2}}$.

30. $\left(\frac{1}{a^3} + \frac{1}{b^3}\right)^{\frac{1}{3}}$.

31. $\left(4 + \frac{m}{n}\right)^{\frac{1}{3}}$.

32. $(x^ny^{2n}z^{3n})^{\frac{1}{n}}$.

33. $\sqrt[m]{a^my^{m+1}z^{m+4}}$.

34. $\sqrt[4]{a^2x^2}$.

35. $\sqrt[6]{216x^3y^6z^9}$.

36. $(49a^{2n}b^8c^4)^{\frac{1}{2}}$.

101. Addition and subtraction of radicals. Two radicals which, when reduced to their simplest forms, have the same order and the same radicand, are said to be **similar.**

Thus, $5\sqrt{2}$ and $-2\sqrt{2}$ are similar; so also are $\sqrt[3]{81}$ and $2b\sqrt[3]{3a^3}$ since $\sqrt[3]{81} = 3\sqrt[3]{3}$ and $2b\sqrt[3]{3a^3} = 2ab\sqrt[3]{3}$.

On the other hand, $\sqrt{2}$ and $\sqrt{3}$ are dissimilar; so are $\sqrt{2}$ and $\sqrt[3]{2}$.

The algebraic sum of similar radicals equals the common radical factor multiplied by the sum of its coefficients.

Example 1. $8\sqrt{2} + 3\sqrt{2} - 4\sqrt{2} = (8 + 3 - 4)\sqrt{2} = 7\sqrt{2}$.

Example 2.
$$\begin{aligned}\sqrt{75} + 3\sqrt{12} - 5\sqrt{27} &= \sqrt{25 \cdot 3} + 3\sqrt{4 \cdot 3} - 5\sqrt{9 \cdot 3} \\ &= 5\sqrt{3} + 6\sqrt{3} - 15\sqrt{3} \\ &= -4\sqrt{3}.\end{aligned}$$

Example 3.
$$\begin{aligned}&\tfrac{1}{2}\sqrt{a^3} + 2b\sqrt{\frac{b^2}{a}} - 3b\sqrt{25a^5b^2} \\ &= \tfrac{1}{2}\sqrt{a^2 \cdot a} + 2b\sqrt{\frac{ab^2}{a^2}} - 3b\sqrt{25a^4b^2a} \\ &= \frac{a}{2}\sqrt{a} + \frac{2b^2}{a}\sqrt{a} - 15a^2b^2\sqrt{a} \\ &= \left(\frac{a}{2} + \frac{2b^2}{a} - 15a^2b^2\right)\sqrt{a}.\end{aligned}$$

EXERCISES

Collect terms and simplify:

1. $3\sqrt{2} + 8\sqrt{2}$.
2. $\sqrt{3} - 2\sqrt{3} + 11\sqrt{3}$.
3. $\sqrt{20} + 8\sqrt{45} - \sqrt{5}$.
4. $3\sqrt{28} - \sqrt{63} + 4\sqrt{175}$.
5. $\sqrt[3]{81} + 5\sqrt[3]{24} - \sqrt[3]{375}$.
6. $\sqrt[3]{16} + 9\sqrt[3]{250} - \sqrt[3]{\frac{1}{4}}$.
7. $6\sqrt{\frac{3}{8}} - \sqrt{24} - \sqrt{\frac{2}{3}} + 8\sqrt{6}$.
8. $\sqrt{a} + 6\sqrt{a} + 2\sqrt{a}$.
9. $3\sqrt{b^3} + 4\sqrt{a^2bc^4} + \sqrt{4b^5c^2}$.
10. $\sqrt[3]{a^2b} - 4\sqrt[3]{8a^5b^4c^3} + 3\sqrt[3]{64a^2b^7}$.
11. $\sqrt[4]{a^4bc^6} + 8\sqrt[4]{b^9c^2} - 5\sqrt[4]{a^8b^5c^2}$.
12. $\sqrt[3]{\frac{1}{b}} + \sqrt[3]{a^3b^2} - \sqrt[3]{27b^5}$.
13. $\sqrt{(a+b)^3} - \sqrt{a^3 + a^2b} - \sqrt{ab^2 + b^3}$.
14. $\sqrt{a} + \sqrt{\frac{1}{a}} + \sqrt{a^3 + 2a^2 + a}$.
15. $\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}$.
16. $3a^{\frac{1}{2}}b + 4a^{\frac{3}{2}}c + 5a^{\frac{1}{2}}d$.
17. $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 5x^{\frac{3}{2}}y^{\frac{3}{2}} + 2x^{\frac{1}{2}}y^{\frac{3}{2}}$.

SOLUTION: Taking out the common monomial factor $x^{\frac{1}{2}}y^{\frac{1}{2}}$, we have $2x^{\frac{1}{2}}y^{\frac{1}{2}} - 5x^{\frac{3}{2}}y^{\frac{3}{2}} + 2x^{\frac{1}{2}}y^{\frac{3}{2}} = (2 - 5xy^2 + 2y)x^{\frac{1}{2}}y^{\frac{1}{2}}$.

18. $3a^{\frac{1}{3}}y^{\frac{1}{3}} - 2a^{\frac{1}{3}}y^{\frac{2}{3}}$.
19. $2^{\frac{1}{2}}a^{\frac{1}{2}}b + 8^{\frac{1}{2}}a^{\frac{1}{2}}b^2 - 18^{\frac{1}{2}}a^{\frac{1}{2}}b^3$.
20. $t^{\frac{1}{2}} + 2t^{\frac{2}{3}} + 3t^{\frac{1}{4}}$.

21. $3^{\frac{1}{2}} + 12^{\frac{1}{2}}a - 27^{\frac{1}{2}}b$.

22. $\sqrt{a^3b^2} + 3\sqrt{4a^3b^4} + 5a\sqrt{9ab^2} - 4ab\sqrt{ab^2}$.

23. $3\sqrt{8} - 4\sqrt{72} + 6\sqrt{48} - \sqrt{108}$.

24. $\sqrt[3]{(x+y)^4} - \sqrt[3]{x^4 + x^3y} - \sqrt[3]{8xy^3 + 8y^4}$.

25. $\sqrt[4]{\frac{x}{y}} + \sqrt[4]{x^5y^3} + \sqrt[4]{xy^7}$.

102. Radicals reduced to the same order. Two radicals are of the same order if they have the same index.

Radicals of different orders may be reduced to the same order. This is most easily done by the use of fractional exponents reduced to a common denominator.

Example 1. Reduce $\sqrt{a}$ and $\sqrt[3]{a}$ to the same order.

SOLUTION:
$$\sqrt{a} = a^{\frac{1}{2}} = a^{\frac{3}{6}} = \sqrt[6]{a^3}.$$
$$\sqrt[3]{a} = a^{\frac{1}{3}} = a^{\frac{2}{6}} = \sqrt[6]{a^2}.$$

Example 2. Reduce $2\sqrt{xy}$ and $5\sqrt[3]{xy^2z^2}$ to the same order.

SOLUTION:
$$2\sqrt{xy} = 2x^{\frac{1}{2}}y^{\frac{1}{2}} = 2x^{\frac{3}{6}}y^{\frac{3}{6}} = 2\sqrt[6]{x^3y^3}.$$
$$5\sqrt[3]{xy^2z^2} = 5x^{\frac{1}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}} = 5x^{\frac{2}{6}}y^{\frac{4}{6}}z^{\frac{4}{6}} = 5\sqrt[6]{x^2y^4z^4}.$$

EXERCISES

Reduce to the same order:

1. $\sqrt{2}$, $\sqrt[3]{4}$.
2. $\sqrt[3]{b}$, $\sqrt[4]{c}$.
3. $\sqrt{xy}$, $2\sqrt[4]{mn}$.
4. $\sqrt[3]{5}$, $\sqrt[5]{7}$.
5. $\sqrt{ab}$, $\sqrt[3]{x^2y}$.
6. $2\sqrt{m}$, $\sqrt[6]{x^2y^5}$.
7. $2\sqrt{2}$, $3\sqrt[3]{3}$, $5\sqrt[5]{5}$.
8. $\sqrt[3]{x^2y^2z}$, $2\sqrt[4]{xy^3z^2}$.
9. $\frac{1}{2}\sqrt[5]{ab^3}$, $4\sqrt[7]{ab^5c^3}$.
10. $14\sqrt[4]{am^2}$, $9\sqrt[6]{a^3m^5}$.

103. Multiplication of radicals. We may well illustrate the process of finding the product of two radicals by examples.

Example 1. Multiply $\sqrt{3}$ by $\sqrt{5}$.

SOLUTION: $\sqrt{3} \cdot \sqrt{5} = 3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = 15^{\frac{1}{2}}$ (Law IV, Art. 89.)

$= \sqrt{15}$.

Example 2. Multiply $2\sqrt[3]{ab^2m}$ by $3\sqrt[3]{mn}$.

SOLUTION: $2\sqrt[3]{ab^2m} \cdot 3\sqrt[3]{mn} = 2(ab^2m)^{\frac{1}{3}} \cdot 3(mn)^{\frac{1}{3}}$

$= 6(ab^2m^2n)^{\frac{1}{3}}$ (Law IV, Art. 89.)

$= 6\sqrt[3]{ab^2m^2n}$.

Radicals of different orders may be reduced to the same order by the methods of the last article.

Example 3. Multiply $2\sqrt{ab}$ by $5\sqrt[3]{a^2b}$.

SOLUTION: $2\sqrt{ab} = 2\sqrt[6]{a^3b^3}$,

$5\sqrt[3]{a^2b} = 5\sqrt[6]{a^4b^2}$,

$2\sqrt[6]{a^3b^3} \cdot 5\sqrt[6]{a^4b^2} = 10\sqrt[6]{a^7b^5} = 10a\sqrt[6]{ab^5}$.

The principle used in finding the product of radicals of the same order may be stated as follows:

The product of two or more radicals of the same order is equal to the product of their coefficients times a radical of the same order whose radicand is the product of the given radicands.

EXERCISES

Perform indicated multiplications and simplify as far as possible:

1. $3\sqrt{m} \cdot 5\sqrt{an}$.
2. $4\sqrt{abc} \cdot a\sqrt{bc}$.
3. $2\sqrt{5} \cdot \sqrt{10} \cdot 4\sqrt{35}$.
4. $\sqrt{x} \cdot \sqrt[3]{y^2}$.
5. $a\sqrt{b} \cdot \sqrt[3]{ab^2}$.
6. $\sqrt{10} \cdot \sqrt[4]{2}$.
7. $2\sqrt{a} \cdot 5\sqrt[3]{a^2b} \cdot \sqrt[4]{a^3c}$.
8. $\sqrt[4]{4} \cdot \sqrt[4]{8}$.
9. $\sqrt[3]{6} \cdot \sqrt[3]{36} \cdot \sqrt[6]{5}$.
10. $\sqrt{5} \cdot \sqrt[3]{\frac{2}{5}}$.

The multiplication of radicals is often much more easily performed by the use of fractional exponents. That method should be used in the next ten exercises.

11. $\sqrt[3]{m} \cdot \sqrt[3]{m^2n} \cdot 5\sqrt[4]{m^3n^2}$.

SOLUTION: $\sqrt[3]{m} \cdot \sqrt[3]{m^2n} \cdot 5\sqrt[4]{m^3n^2} = m^{\frac{1}{3}} m^{\frac{2}{3}} n^{\frac{1}{3}} 5 m^{\frac{3}{4}} n^{\frac{1}{2}}$

$= 5 m^{\frac{1}{1}} n^{\frac{1}{2}} = 5m \cdot m^{\frac{1}{1}} n^{\frac{1}{2}} = 5m \cdot m^{\frac{12}{12}} n^{\frac{6}{12}}$

$= 5m\sqrt[12]{m^9 n^{10}}$.

12. $\sqrt{xy} \cdot 2\sqrt[3]{x^2y^2}$.

13. $\sqrt{a} : \sqrt{b} \cdot \sqrt[4]{a^2b^3}$.

14. $\sqrt{3} \cdot \sqrt[3]{3}$.

15. $3\sqrt{5} \cdot \sqrt[4]{2}$.

16. $\sqrt[5]{128} \cdot \sqrt[3]{500}$.

17. $\sqrt[5]{x^2y^3z^4} \cdot \sqrt[4]{x^3y^2z}$.

18. $3a^{\frac{1}{2}}c^{\frac{1}{3}} \cdot 5a^{\frac{1}{3}}bc$.

19. $-8a^{\frac{1}{3}}b^{\frac{1}{4}} \cdot -5a^{\frac{1}{2}}bc^2$.

20. $\frac{2}{3}a^{\frac{1}{2}}b^2c^{\frac{1}{3}} \cdot 9a^{\frac{1}{2}}c^3$.

21. $-r^{\frac{1}{2}}s^{\frac{1}{3}} \cdot 3r^2sp$.

22. $(\sqrt{a} + 2\sqrt{b})(3\sqrt{a} - 5\sqrt{b})$.

SOLUTION:

$$\begin{array}{r} \sqrt{a} + 2\sqrt{b} \\ \underline{3\sqrt{a} - 5\sqrt{b}} \\ 3a + 6\sqrt{ab} \\ \quad - 5\sqrt{ab} - 10b \\ \hline 3a + \sqrt{ab} - 10b. \end{array}$$

23. $(\sqrt{3} - \sqrt{5})(\sqrt{3} + \sqrt{5})$.

24. $(\sqrt{m} + \sqrt{n})^2$.

25. $2\sqrt{3}(\sqrt{3} + \sqrt{5} - 4\sqrt{6})$.

26. $2a\sqrt{b}(ab\sqrt{b} - a\sqrt{ab} + 5\sqrt{a})$.

27. $(5 - 2\sqrt{5})(3 - \sqrt{5})$.

28. $(\sqrt{7} + \sqrt{11})(\sqrt{3} - \sqrt{5})$.

29. $(\sqrt{2} + 2\sqrt{3} - 3\sqrt{5})^2$.

30. $(a^{\frac{1}{2}}b^{\frac{1}{2}} - c^{\frac{1}{2}})^2$.

31. $(3a^{\frac{1}{2}} + 2b^{\frac{1}{2}})^2$.

32. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^3$.

33. $(2\sqrt{7} - 8\sqrt{28} - \sqrt{63})^2$.

34. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}})$.

35. $(\sqrt{5} + 4\sqrt{6} - 2\sqrt{3})^2$.

36. Find the value of $x^2 - 4x + 1$ if $x = 2 + \sqrt{3}$.

37. Find the value of $3x^2 + 4x - 2$ if $x = \dfrac{-2 - \sqrt{10}}{3}$.

38. Find the value of $ax^2 + bx + c$ if $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$.

104. Division of radicals. When the dividend and divisor are monomials of the same order the division may be performed by the use of the principle

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$

Example 1. $\dfrac{6\sqrt{6}}{2\sqrt{3}} = 3\sqrt{\dfrac{6}{3}} = 3\sqrt{2}.$

Example 2. $\dfrac{\sqrt{28}}{\sqrt{12}} = \sqrt{\dfrac{28}{12}} = \sqrt{\dfrac{7}{3}} = \dfrac{1}{3}\sqrt{21}.$

For purposes of computation it is usually desirable that the denominator of the quotient be made rational.

Example 3. $\dfrac{\sqrt{10}}{\sqrt{7}} = \dfrac{\sqrt{10}\ \sqrt{7}}{\sqrt{7}\ \sqrt{7}} = \dfrac{\sqrt{70}}{7}.$

Example 4. $\dfrac{\sqrt[3]{14}}{\sqrt[3]{36}} = \dfrac{\sqrt[3]{14} \cdot \sqrt[3]{6}}{\sqrt[3]{36}\ \sqrt[3]{6}} = \dfrac{\sqrt[3]{84}}{\sqrt[3]{216}} = \dfrac{\sqrt[3]{84}}{6}.$

Example 5. $\dfrac{3\sqrt{5} + \sqrt{2}}{2\sqrt{5} - 3\sqrt{2}} = \dfrac{(3\sqrt{5} + \sqrt{2})\ (2\sqrt{5} + 3\sqrt{2})}{(2\sqrt{5} - 3\sqrt{2})\ (2\sqrt{5} + 3\sqrt{2})} =$

$$\frac{30 + 2\sqrt{10} + 9\sqrt{10} + 6}{20 - 18} = \frac{36 + 11\sqrt{10}}{2}.$$

EXERCISES

Perform the following divisions, obtaining results with rational denominators.

1. $\sqrt{6} \div \sqrt{3}$.
2. $\sqrt{80} \div \sqrt{5}$.
3. $\sqrt{42} \div \sqrt{6}$.
4. $\sqrt[3]{35} \div \sqrt[3]{5}$.
5. $\sqrt[3]{135} \div \sqrt[3]{5}$.
6. $\sqrt{ab} \div \sqrt{a}$.
7. $\sqrt[3]{a^2bc^2} \div \sqrt[3]{ac^2}$.
8. $6\sqrt{5} \div 2$.
9. $6\sqrt{5} \div 2\sqrt{5}$.
10. $\sqrt{\frac{1}{2}} \div \sqrt{\frac{2}{3}}$.
11. $\sqrt[3]{ab} \div \sqrt[3]{cd}$.
12. $\sqrt[3]{a^2m} \div \sqrt[3]{am^2}$.
13. $\sqrt[3]{x} \div \sqrt[3]{y}$.
14. $6\sqrt{150} \div 5\sqrt{45}$.
15. $(\sqrt{12} - 4\sqrt{6}) \div \sqrt{2}$.
16. $(\sqrt{10} + 3\sqrt{15} - 7\sqrt{35}) \div \sqrt{10}$.
17. $6 \div 2\sqrt[3]{7}$.
18. $7 \div 2\sqrt[4]{5}$.
19. $\sqrt{\frac{a}{b}} \div \sqrt{ab}$.
20. $(\sqrt{a} + \sqrt{b}) \div \sqrt{ab}$.
21. $(\sqrt{a} + \sqrt{b}) \div \sqrt{a+b}$.
22. $(3 + \sqrt{6}) \div (2\sqrt{6} - 1)$.
23. $1 \div (2\sqrt{3} - 4\sqrt{5})$.
24. $(5\sqrt{7} - 3\sqrt{11}) \div (\sqrt{7} + 4\sqrt{11})$.
25. $7\sqrt{2} \div (2\sqrt{5} - 9\sqrt{2})$.
26. $\dfrac{3}{3 + \sqrt{6}}$.
27. $\dfrac{2}{2 - \sqrt{2}}$.
28. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$.
29. $\dfrac{5\sqrt{3} - 3\sqrt{5}}{\sqrt{5} - \sqrt{3}}$.

PROBLEMS

(Reduce the answers to the simplest radical form.)

1. What is the diagonal of a square whose side is s?
2. What is the diagonal of a square whose area is 280 square inches?

3. Express the volume V of a cube in terms of d, the diagonal of one face. Also express d in terms of V.

4. Find the altitude and the area of an equilateral triangle whose side is s.

5. Find the side s of an equilateral triangle whose area is A.

6. Find the side of an equilateral triangle whose area is 420 square inches.

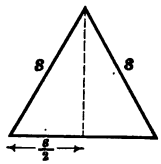

FIG. 44

7. If S is the surface and r the radius of a sphere, then $S = 4\pi r^2$. Find r if $S = 2000$ square inches.

8. A regular hexagon can be divided into 6 equilateral triangles. Find the area of a regular hexagon whose side is 10 inches. (Fig. 45.)

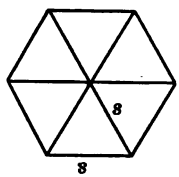

FIG. 45

9. The volume V of a pyramid of base area b and altitude h is given by the formula $V = \frac{bh}{3}$. A pyramid with a volume of 300 cubic feet and an altitude of 9 feet, has a square base. Find the length of the side of the base.

10. Find the altitude and the volume of a regular pyramid with a square base, if the side of the base is s, and the lateral edge $\frac{3}{2}s$.

In a regular pyramid, the perpendicular from the vertex to the base meets the base at its center. In Fig. 46, OA is a lateral edge.

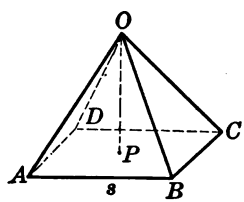

FIG. 46

11. Find the altitude and volume in the last problem if $s = 10$.

12. Show that the total surface of the pyramid in Problem 10 is $s^2(1 + 2\sqrt{2})$.

13. From the formula of Problem 12 find the total surface if $s = 10$.

14. The pyramid in Fig. 47 is a regular tetrahedron. If each edge is 10, find AD, AP, and lastly the altitude of the pyramid OP.

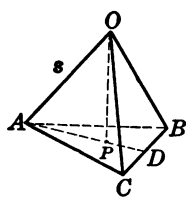

FIG. 47

The faces of a regular tetrahedron are equilateral triangles. In Fig. 47, $AP = \frac{3}{4}AD$, where AD is an altitude of the triangle ABC.

15. Find the volume of a regular tetrahedron whose edge is 10.

105. Square roots of polynomials. The following solution will illustrate the process of square root.

Find the square root of $9x^4 + 2x^3 + 22x^2 + 12x + 9$.

SOLUTION:

$$\begin{array}{lll} & 9x^4 + 12x^3 + 22x^2 + 12x + 9 & \mid 3x^2 + 2x + 3 \\ & 9x^4 & \\ \hline 6x^2 + 2x & 12x^3 + 22x^2 & \\ & 12x^3 + 4x^2 & \\ \hline 6x^2 + 4x + 3 & 18x^2 + 12x + 9 & \\ & 18x^2 + 12x + 9 & \end{array}$$

EXPLANATION. (1.) The first term of the root is $\sqrt{9x^4} = 3x^2$.

(2) The trial divisor is $2(3x^2) = 6x^2$.

(3) The second term of the root is $12x^3 \div 6x^2 = 2x$.

(4) The complete divisor is $6x^2 + 2x$.

(5) The second trial divisor is $2(3x^2 + 2x) = 6x^2 + 4x$.

(6) The third term of the root is $18x^2 \div 6x^2 = 3$.

(7) The second complete divisor is $6x^2 + 4x + 3$.

EXERCISES

Find the square roots of the following:

1. $25x^2 + 60xy + 36y^2$.
2. $4a^4 - 20a^2b + 25b^2$.
3. $a^4 - 2a^3 + 3a^2 - 2a + 1$.
4. $4x^4 + 20x^3 + 29x^2 + 10x + 1$.
5. $a^2 + 4b^2 + 9c^2 - 4ab + 6ac - 12bc$.
6. $x^2 + \frac{1}{9}y^2 + \frac{2}{5}x + \frac{1}{25} - \frac{2}{3}xy - \frac{2}{15}y$.
7. $x^6 - 2x^5 + 7x^4 - 2x^3 + 5x^2 + 12x + 4$.
8. $a^{12} - 4a^9 + 4a^6 - a^6 + 2a^3 + \frac{1}{4}$.

106. Square roots of numbers expressed in Arabic symbols. The following solution will illustrate the process of extracting the square root of a number expressed in the Arabic notation.

Find the square root of 419904.

SOLUTION:

	41'99'04	648
	360000	
Trial divisor = $2 \times 600 = 1200$	59904	
Complete divisor = $1200 + 40 = 1240$	49600	
Trial divisor = $2 \times 640 = 1280$	10304	
Complete divisor = $1280 + 8 = 1288$	10304	

EXERCISES

Find the square roots of the following. In each of the Exercises 5–12, find the root correct to two decimal places.

1. 2116.
2. 6084.
3. 524176.
4. 822649.
5. 85.
6. 3.
7. .1.
8. .016.
9. 3.1416.
10. $\frac{25}{49}$.
11. $\frac{5}{7}$.
12. $\frac{1}{2}$.

107. Square roots of radical expressions of the form $a + \sqrt{b}$. The square of $\sqrt{x} \pm \sqrt{y}$ is $x + y \pm 2\sqrt{xy}$. Hence if $a + \sqrt{b}$ can be put into the form of $x + y \pm 2\sqrt{xy}$ its square root can be found.

Example. Find the square root of $7 + \sqrt{48}$.

SOLUTION: $7 + \sqrt{48} = 7 + 2\sqrt{12} = 4 + 3 + 2\sqrt{4 \cdot 3}$.

Hence, $\sqrt{7 + \sqrt{48}} = 2 + \sqrt{3}$.

EXERCISES

Find the square root of each of the following:

1. $3 - 2\sqrt{2}$.
2. $6 + 2\sqrt{5}$.
3. $11 + 6\sqrt{2}$.
4. $10 + \sqrt{96}$.
5. $151 + 24\sqrt{7}$.
6. $27 - 10\sqrt{2}$.

***108. Cube roots of polynomials.** In a first course in algebra the process of finding the square root is explained by the aid of the formula

$$(a + b)^2 = a^2 + 2ab + b^2.$$

Similarly the method of finding the cube root is obtained by studying the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3. \qquad (1)$$

Example 1. Find the cube root of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

SOLUTION: The work may be put in the following form.

$$8x^3 + 36x^2y + 54xy^2 + 27y^3 \mid 2x + 3y.$$

$$8x^3$$

Trial divisor $= 3(2x)^2 = 12x^2$	$36x^2y + 54xy^2 + 27y^3$
Complete divisor $= 12x^2 + 3(2x)(3y) + (3y)^2$	
$= 12x^2 + 18xy + 9y^2$	$36x^2y + 54xy^2 + 27y^3$

EXPLANATION. In (1) the first term of the cube root is $\sqrt[3]{a^3} = a$. Similarly in the present example, the first term of the cube root is $\sqrt[3]{8x^3} = 2x$.

In (1) the second term of the cube root, b, may be obtained by dividing $3a^2b$ by 3 times the square of the root already found, which is $3a^2$, and which is called the trial divisor. In the present example, the trial divisor is $3(2x)^2 = 12x^2$. From $36x^2y \div 12x^2 = 3y$, we find the second term of the root.

The remainder in (1), after a^3 is subtracted, may be written as the product of two factors, or $3a^2b + 3ab^2 + b^3 = b(3a^2 + 3ab + b^2)$. In the present example we need to find an expression of the form $3a^2 + 3ab + b^2$ where $a = 2x$, and $b = 3y$. This gives

$$3(2x)^2 + 3(2x)(3y) + (3y)^2 = 12x^2 + 18xy + 9y^2,$$

which is the complete divisor. Then

$$3y(12x^2 + 18xy + 9y^2) \quad = 36x^2y + 54xy^2 + 27y^3.$$

* Articles 108 and 109 may well be regarded as supplementary work.

When this product is subtracted the remainder is zero. The process is then complete and the cube root is $2x + 3y$.

It should be noticed that the complete divisor is formed by adding to the trial divisor an expression of the form $3ab + b^2$.

Example 2. Find the cube root of

$$8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1.$$

SOLUTION:

$8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$	$2x^2 - 3x + 1$
$8x^6$	
1st trial divisor, $3(2x^2)^2 = 12x^4$	$-36x^5 + 66x^4 - 63x^3$
1st complete divisor, $12x^4 - 18x^3 + 9x^2$	$-36x^5 + 54x^4 - 27x^3$
2nd trial divisor, $3(2x^2 - 3x)^2 = 12x^4 - 36x^3 + 27x^2$	$12x^4 - 36x^3 + 33x^2 - 9x + 1$
2nd complete divisor, $12x^4 - 36x^3 + 33x^2 - 9x + 1$	$12x^4 - 36x^3 + 33x^2 - 9x + 1$

The first two terms of the root are found as in Example 1.

To find the third term we proceed as in finding the second term.

Three times the square of the part of the root already found is

$$3(2x^2 - 3x)^2 = 12x^4 - 36x^3 + 27x^2,$$

which is the trial divisor. Dividing, we find the next term in the root to be 1.

To find the complete divisor we must add to the trial divisor

$$3(2x^2 - 3x) \cdot 1 + 1^2 = 6x^2 - 9x + 1.$$

The complete divisor is then

$$12x^4 - 36x^3 + 33x^2 - 9x + 1.$$

Multiplying this by the last term of the root and subtracting, the remainder is zero. The process is then complete and the cube root is $2x^2 - 3x + 1$.

If the root contains more than three terms, the succeeding terms can be found by continuing the process used in finding the second and third terms.

EXERCISES

Find the cube roots of the following expressions:

1. $27x^3 + 27x^2 + 9x + 1.$
2. $x^3 + 6x^2y + 12xy^2 + 8y^3.$
3. $8a^3 - 36a^2b + 54ab^2 - 27b^3.$

4. $p^6 - 3p^4y^3 + 3p^2y^6 - y^9$.

5. $64a^3b^3 + 144a^2b^2 + 108ab + 27$.

6. $1 - 15m^2x + 75m^4x^2 - 125m^6x^3$.

7. $216x^6y^3 + 108m^2x^4y^2 + 18m^4x^2y + m^6$.

8. $m^3 + \frac{3}{2}m^2 + \frac{3}{4}m + \frac{1}{8}$.

9. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$.

10. $b^6 - 3b^5 + 5b^3 - 3b - 1$.

11. $8x^{12} - 36x^{10} + 66x^8 - 63x^6 + 33x^4 - 9x^2 + 1$.

12. $125x^3 + 150x^2y + 225x^2 + 60xy^2 + 180xy + 135x + 8y^3 + 36y^2 + 54y + 27$.

*** 109. Cube roots of numbers expressed in Arabic symbols.** The process used in finding the cube roots of polynomials can be applied to numbers expressed in Arabic symbols. This will be illustrated by an example.

Example. Find the cube root of 185193.

SOLUTION:

	$185'193$ $\vert\underline{57}$	$t = 50,$
	$125000 = t^3$	$u = 7.$
Trial divisor $= 3t^2 = 3 \cdot 50^2 =$	7500	60193 contains $3t^2u + 3tu^2 + u^3$.
$3tu = 3 \cdot 50 \cdot 7 =$	1050	
$u^2 = 7^2 =$	$\underline{49}$	
Complete divisor $=$		
$3t^2 + 3tu + u^2 =$	8599	$60193 = 3t^2u + 3tu^2 + u^3$.

EXPLANATION. (1) The given number is divided into periods of 3 figures each to determine the number of orders in the cube root.

(2) Since the number contains 2 such periods, the root contains two orders, tens and units, which may be represented by t and u, respectively. Hence the cube is in the form

$$(t + u)^3 = t^3 + 3t^2u + 3tu^2 + u^3.$$

(3) The largest cube of tens contained in the number is

$$t^3 = 50^3 = 125000.$$

(4) After t^3 is subtracted the remainder contains $3t^2u + 3tu^2 + u^3$.

* Finding cube roots of numbers may well be delayed until the next chapter where it is done much more simply and easily by the aid of logarithms.

(5) The trial divisor is 3 times the square of the root already found, which is $3t^2 = 3 \cdot 50^2 = 7500$.

(6) The next figure in the root is found by division to be 7.

(7) The complete divisor is

$$3t^2 + 3tu + u^2 = 3 \cdot 50^2 + 3 \cdot 50 \cdot 7 + 7^2 = 8599.$$

(8) Multiplying the complete divisor by the new figure in the root we have $7 \times 8599 = 60193$. Subtracting, the remainder is zero. Hence the process is complete, and the cube root is 57.

(9) If the cube root contains more than two orders, the remaining orders may be found by repeating steps 5, 6, 7, and 8.

EXERCISES

Find the cube roots of the following:

1. 13824.
2. 110592.
3. 421875.
4. 804357.
5. 636056.
6. 21952.
7. 132651.
8. 4913.

MISCELLANEOUS EXERCISES AND PROBLEMS

In Exercises 1–6 rationalize the denominator and find the values of the fractions correct to two decimal places.

1. $\dfrac{3}{1+\sqrt{2}}$.
2. $\dfrac{5}{\sqrt{3}+\sqrt{5}}$.
3. $\dfrac{5+\sqrt{3}}{2-2\sqrt{3}}$.
4. $\dfrac{2\sqrt{5}-\sqrt{7}}{\sqrt{5}+3\sqrt{7}}$.
5. $\dfrac{2+\sqrt{5}}{3\sqrt{7}}$.
6. $\dfrac{8}{5\sqrt{2}-4\sqrt{3}}$.
7. Find the value of $x^2 - 6x + 2$ when $x = 3 + 2\sqrt{2}$.
8. Find the value of $x^2 - 4x - 1$ when $x = 2 - \sqrt{5}$.
9. Find the value of $3x^2 - 2x - 6$ when $x = 1 + \dfrac{\sqrt{19}}{3}$.
10. The area of a square is $14 + 6\sqrt{5}$. Find the side correct to two decimal places.

11. Find the side of a square whose area is 998001 square feet.

12. Find correct to two decimal places the radius of a circle whose area is 50 square inches.

13. In Fig. 48, T is a right triangle, and T_1, T_2, and T_3 are equilateral. Prove that $T_1 = T_2 + T_3$.

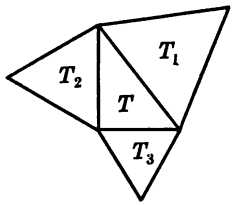

FIG. 48

14. The circumference of a circle equals the perimeter of a square which is $4s$. Find the difference in their areas in terms of s.

15. Assume a given unit of length and construct geometrically $\sqrt{5}$.

SUGGESTION. Use coördinate paper. Construct a right triangle with legs 1 and 2. Then the hypotenuse equals $\sqrt{1+4} = \sqrt{5}$. By actual measurement the value of $\sqrt{5}$ can be determined correct to tenths, or even hundredths, if the scale is taken sufficiently large.

16. Construct the following square roots: $\sqrt{2}$, $\sqrt{10}$, $\sqrt{17}$, $\sqrt{3}$, $\sqrt{20}$.

SUGGESTION. In constructing $\sqrt{3}$ construct a triangle with base 1 and altitude $\sqrt{2}$.

17. Plot carefully the graph of $y^2 = x$ for values of y from 1 to 10. Solve the equation for y. This gives $y = \sqrt{x}$. If now for given values of x we read off from the curve the values of y, we thus obtain the square roots of those values of x. By the aid of this curve find the approximate value of the square root of 2, 3, 10, 16, 38, and 75.

18. Similarly from the curve $y = x^3$ plotted from $x = 1$ to $x = 5$ obtain the cube root of 2, 3, 27, 40, 65.

CHAPTER XII

LOGARITHMS

110. Definition of a logarithm. In the expression $10^3 = 1000$, the exponent 3 is called the **logarithm** of 1000 to the base 10, and is written $3 = \log_{10}1000$.

In general, if $a^x = y$, $(a>0,\ a \neq 1)$, then x is said to be the logarithm of y to the base a, which is written $x = \log_a y$.

The two equations

$$a^x = y,$$

and

$$x = \log_a y$$

are thus equivalent. That is, the logarithm is an exponent.

Without using symbols, the logarithm of a given number may be defined as the exponent of the power to which a number called the base must be raised to equal the given number.

In what follows, we shall assume that the laws of exponents (Arts. 89–94), which apply to rational exponents, are valid for any set of exponents with which we are concerned.

111. Numbers expressed as powers of given numbers. We know how to express many numbers as powers of a given number. These same relations may be expressed by means of logarithms.

Thus, $16 = 4^2$, and $2 = \log_4 16$; $8 = 4^{\frac{3}{2}}$, and $\frac{3}{2} = \log_4 8$; $\frac{1}{64} = 4^{-3}$, and $-3 = \log_4 \frac{1}{64}$; $100 = 10^2$, and $2 = \log_{10} 100$; $\frac{1}{10} = 10^{-1}$, and $-1 = \log_{10} \frac{1}{10}$.

EXERCISES

1. $\log_6 36 = ?$ $\log_{10} 100 = ?$ $\log_9 3 = ?$ $\log_2 128 = ?$ $\log_3 81 = ?$

2. Fill out the following table:

Base	Number	Logarithm
10	10	
	81	4
$\frac{1}{2}$	8	
8	$\frac{1}{2}$	
10		-3
10	1	

3. Write the following in the notation of logarithms: (*a*) $452 = 10^{1.6551}$; (*b*) $5.1 = 10^{.7076}$; (*c*) $6470 = 10^{3.8109}$; (*d*) $24.59 = 10^{1.3907}$.

4. Write the following using exponents: (*a*) $\log_{10} 2 = .3010$; (*b*) $\log_{10} 99 = 1.9956$; (*c*) $\log_{10} 34.2 = 1.5340$; (*d*) $\log_{10} 7000 = 3.8451$.

112. Characteristic and mantissa. The integral part of a logarithm is called the **characteristic** and the decimal part the **mantissa.**

Thus, in $\log_{10} 245 = 2.3892$, the characteristic is 2 and the mantissa is .3892.

The characteristic may be negative.

Thus, $\log_{10} .01 = -2$.

It is usually convenient to keep the mantissa positive even though the logarithm is negative.

Thus, $\log_{10} \frac{1}{5} = -.6990$. But this may be written $\bar{1}.3010$, which has a positive mantissa, and a characteristic -1. The minus sign is placed over the 1 to indicate that the characteristic alone is negative.

113. Logarithm of a product. *The logarithm of a product equals the sum of the logarithms of its factors.*

PROOF: Let $\log_a u = x$ and $\log_a v = y$,

then $a^x = u$, $a^y = v$, (Definition of logarithm)

and $uv = a^{x+y}$. (Art. 89)

Hence, $\log_a uv = x + y$,

that is, $\log_a uv = \log_a u + \log_a v$.

Similarly, $\log_a(uvw) = \log_a u + \log_a v + \log_a w$,
and so on for any number of factors.

Example. $\log_{10}(79 \times 642) = \log_{10}79 + \log_{10}642$.

114. Logarithm of a quotient. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

PROOF: As above, let $\log_a u = x$ and $\log_a v = y$.

Then $a^x = u$, and $a^y = v$,

and $\frac{u}{v} = a^{x-y}$.

Hence, $\log_a \frac{u}{v} = x - y$,

that is, $\log_a \frac{u}{v} = \log_a u - \log_a v$.

Example. $\log_{10}\frac{245}{912} = \log_{10}245 - \log_{10}912$.

115. Logarithm of a power. *The logarithm of u^v is equal to v times the logarithm of u.*

PROOF: Let $x = \log_a u$, or $a^x = u$.

Then $u^v = (a^x)^v = a^{vx}$ (Art. 89)

Hence, $\log_a u^v = vx = v \log_a u$.

Example. $\log_{10}(455)^{\frac{1}{2}} = \frac{1}{2}\log_{10}455$.

Making $v = n$, and then $v = \frac{1}{n}$, we have respectively,

(*a*) *The logarithm of the nth power of a number is n times the logarithm of the number.*

(*b*) *The logarithm of the real positive nth root of a number is the logarithm of the number divided by n.*

EXERCISES

Express the logarithms * of the following expressions in terms of the logarithms of integers:

1. $\log \dfrac{17\sqrt[3]{2}}{7^{\frac{3}{4}}11^{\frac{1}{4}}}$.

SOLUTION:

$$\log \frac{17\sqrt[3]{12}}{7^{\frac{3}{4}}11^{\frac{1}{4}}} = \log 17 + \log \sqrt[3]{12} - \log 7^{\frac{3}{4}} - \log 11^{\frac{1}{4}} \text{ (Arts. 113, 114, 115)}$$
$$= \log 17 + \tfrac{1}{3}\log 12 - \tfrac{3}{4}\log 7 - \tfrac{1}{4}\log 11 \text{ (Art. 115).}$$

2. $\log \dfrac{48}{95}$.

3. $\log \dfrac{2\sqrt{180}}{19}$.

4. $\log \dfrac{12^2 \cdot 325}{\sqrt[3]{68}}$.

5. $\log \dfrac{(58)^{\frac{1}{3}}}{11\sqrt[4]{2}}$.

6. $\log \dfrac{3\sqrt{5}}{\sqrt[3]{9}}$.

7. $\log \dfrac{5^4 \cdot \sqrt[4]{1.9}}{6 \cdot 11^2}$.

8. $\log \sqrt{\dfrac{245}{311}}$.

9. $\log \left(\sqrt{\dfrac{5}{7}} \cdot \sqrt{\dfrac{3}{8}}\right)$.

10. $\log \left(\sqrt{\dfrac{1}{2}} \cdot \left(\dfrac{7}{43}\right)^{\frac{5}{6}}\right)$.

* When in a problem the same base is used throughout, it is customary not to write the base.

Express the logarithms of the following expressions in terms of the logarithms of prime factors:

11. $\log \dfrac{ab^3}{c\sqrt{d}}$.

SOLUTION: $\log \dfrac{ab^3}{c\sqrt{d}} = \log a + 3\log b - \log c - \frac{1}{2}\log d$.

12. $\log \dfrac{ab}{c^2}$.

13. $\log \dfrac{m\sqrt{n}}{n(m-2)}$.

14. $\log \sqrt{\dfrac{cd}{c-d}}$.

15. $\log(a^2 - b^2)$.

16. $\log \dfrac{1}{\sqrt{c(ax+b)}}$.

17. $\log \dfrac{7x^2}{\sqrt{y}}$.

18. $\log(a^3 + b^3)$.

Express each of the following sums and differences as the logarithm of a single expression:

19. $\log m - \frac{1}{3}\log n + 2\log x$.

SOLUTION: $\log m - \frac{1}{3}\log n + 2\log x = \log m - \log \sqrt[3]{n} + \log x^2 = \log \dfrac{mx^2}{\sqrt[3]{n}}$.

20. $5\log 3 + \frac{2}{3}\log 7 - \frac{1}{2}\log 95$.

21. $\log 48 + 3\log a - 2\log b - \frac{1}{2}\log 15$.

22. $a\log x - b\log y$.

23. $\log(a-b) + \log(a+b)$.

24. $\log 1 - \log 15$.

25. $\frac{1}{2}\log(a^2 + b^2) - 2\log a - 2\log b$.

116. Common logarithms. As 10 is the base of our number system, it is convenient to take 10 as the base of the logarithms used in ordinary computation. These logarithms are known as the **common logarithms,** or the Briggs system of logarithms.

In the following discussion of common logarithms, $\log x$ is written as an abbreviation of $\log_{10} x$.

117. Determination of the characteristic of a common logarithm.

Since $10^3 = 1000$, $\log 1000 = 3$;
" $10^2 = 100$, $\log 100 = 2$;
" $10^1 = 10$, $\log 10 = 1$;
" $10^0 = 1$, $\log 1 = 0$;
" $10^{-1} = .1$, $\log .1 = -1$;
" $10^{-2} = .01$, $\log .01 = -2$;
" $10^{-3} = .001$, $\log .001 = -3$;

and so on.

So far as these powers of 10 are concerned it may be observed that when a number increases, its logarithm increases. Assuming this to be true for all positive numbers we have, for example,

$$\log 100 < \log 400 < \log 1000,$$

or

$$2 < \log 400 < 3.$$

Hence the characteristic of log 400 is 2.

Similarly, $\log .01 < \log .021 < \log .1$,

or

$$-2 < \log .021 < -1.$$

Hence the characteristic of log.021 is -2, as the mantissa is always taken positive.

Exercise. Find the characteristic of the logarithm of each of the following: 15, 348, 8000, 965, .3, .045, .00961, .002.

The above illustrations show that if a number is between 1 and 10 the characteristic of its logarithm is 0; if between 10 and 100, the characteristic is 1; if between 100 and 1000 the characteristic is 2; and so on. Hence, we have the general rule:

The characteristic of the logarithm of a number greater than 1 is one less than the number of digits in the integral part of the number.

Thus, the characteristic of log 34.5 is 1; of log 8964.701 is 3; and so on.

We see also that if a number is between 1 and .1, its characteristic is -1; if between .1 and .01, its characteristic is -2; if between .01 and .001, its characteristic is -3; and so on. Hence, we have the general rule:

The characteristic of the logarithm of a number between 1 and 0 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.

Thus, the characteristic of log.0345 is -2; of log.000479 is -4; and so on.

118. Mantissa of a common logarithm. From the table, p. 187, we shall see later how to find that

$$\log 5682 = 3.7545.$$

Then by Art. 114, $\log 568.2 = \log \frac{5682}{10} = \log 5682 - \log 10$

$$= 3.7545 - 1 = 2.7545.$$

Similarly,
$$\log 56.82 = 2.7545 - 1 = 1.7545,$$
$$\log 5.682 = 1.7545 - 1 = .7545,$$
$$\log .5682 = .7545 - 1 = \bar{1}.7545,$$
$$\log .05682 = \bar{1}.7545 - 1 = \bar{2}.7545.$$

Hence, *the mantissa of the common logarithm of a number is independent of the position of the decimal point in the number.* In other words the common logarithms of numbers which contain the same sequence of figures differ only in their characteristics. Hence tables of common logarithms contain only the mantissas, and the computer must find the characteristics by the rules of Art. 117.

For example, the table, p. 187, gives .8722 as the mantissa for the sequence of figures 745. We know then immediately that log 745 = 2.8722, log 7.45 = .8722 and log 00745 = $\bar{3}$.8722.

119. Graph of $y = \log_a x$ $(a>1)$. A general notion of the values of logarithms of numbers can be easily fixed by reference to the graph of $y = \log_a x$. This graph is also the graph of

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8917
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

$x = a^y$. In making the graph (Fig. 49) we took $a = 10$, but the general form of the graph would not be changed if a were given any other positive value greater than one. In particular, the curve crosses the x-axis at a point a distance 1 from the origin, for any value of a. If the student retains this picture, he should find it easy to keep in mind the following facts when the base is greater than unity.

(1) A negative number does not have a real number for its logarithm.

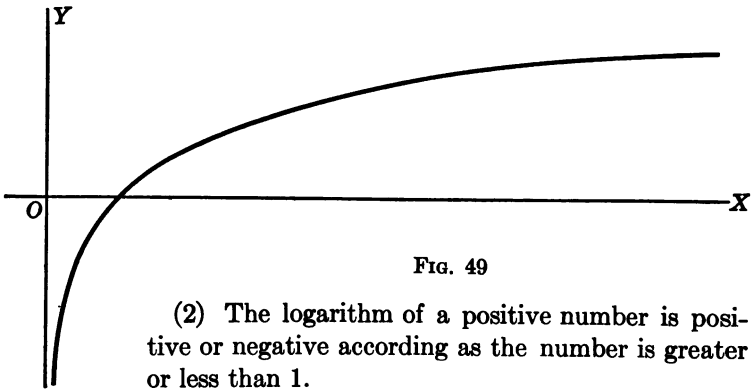

FIG. 49

(2) The logarithm of a positive number is positive or negative according as the number is greater or less than 1.

(3) If x approaches zero, $\log x$ decreases without limit.

(4) If x increases indefinitely, $\log x$ increases without limit.

120. To find from the table the logarithm of a given number.

EXAMPLES

1. Find the logarithm of 821.

Glance down the column headed N for the first two significant figures, then at the top of the table for the third figure. In the row with 82 and the column with 1 is found 9143.

Hence, $\log 821 = 2.9143$.

2. Find the logarithm of 68.42.

This number has more than three significant figures, so that its logarithm is not recorded in the table. It may, however, be obtained approximately

from logarithms recorded in the table by a process called **interpolation.** In our use of this process, it is assumed that to a small change in the number, there corresponds a change in the logarithm which is proportional to the change in the number. This assumption is called the **principle of proportional parts.** As in Example 1, we find that the mantissas of 6840 and 6850 are 8351 and 8357, respectively. The difference between these two mantissas is 6. Since 8642 is two tenths of the interval from 6840 to 6850, by the principle of proportional parts, we add to 8351,

$$0.2 \times 6 = 1+.$$

Hence, $\log 68.42 = 1.8352.$

3. Find the logarithm of .0478.

From the table the mantissa is 6794. Hence the logarithm is $\bar{2}.6794$.

For purposes of computation it is convenient to write this logarithm in the form $8.6794 - 10$. This form will be used hereafter.

121. To find from the table the number that corresponds to a given logarithm.

EXAMPLES

1. Find the number whose logarithm is 2.4675. The mantissa 4675 is not recorded in the table, but it lies between the two adjacent mantissas 4669 and 4683 of the table. The mantissa 4669 corresponds to the number 293 and 4683 corresponds to 294. The number 4675 is $\frac{6}{14}$ of the interval from 4669 to 4683. By the principle of proportional parts, the sequence of figures whose mantissa is 4675 is $2930 + \frac{6}{14} \times 10 = 2934+$.

Hence, $2.4675 = \log 293.4$.

2. Find the number whose logarithm is $9.3025 - 10$.

From the table,

$$\log 0.2000 = 9.3010 - 10$$
$$\log 0.2010 = \underline{9.3032 - 10}$$
$$\text{Difference} = 0.0022$$

$$(9.3025 - 10) - (9.3010 - 10) = 0.0015$$

By the principle of proportional parts, the number is

$$0.2000 + \frac{15}{22} \times 0.0010 = 0.2007.$$

It should be noted that whenever a computation is performed by means of four-place logarithms, the answer should not be given to more than four significant figures.

EXERCISES

Find, from the table, the common logarithms of the following:

1. 367.	**5.** .00289.	**9.** 325000.
2. 190.	**6.** 3642.	**10.** 1.011.
3. 35.	**7.** 97.61.	**11.** .009793.
4. 6.25.	**8.** .1414.	**12.** 68.07

Find, by means of the table, the numbers whose common logarithms are the following:

13. 1.5502.	**17.** 1.3478.	**21.** .9762.
14. 3.9031.	**18.** 8.7542 − 10.	**22.** 4.1572.
15. 9.7348 − 10.	**19.** 3.5740.	**23.** 9.6990 − 10.
16. .8882.	**20.** 7.2182 − 10.	**24.** .4971.

122. Computation by means of logarithms. The application of logarithms to shorten calculations depends upon the properties of logarithms given in Arts. 113–115. By means of logarithms laborious multiplications can be replaced by additions and subtractions; and involution and evolution can be replaced by multiplication and division.

EXAMPLES

1. Find the value of $N = \dfrac{850 \times 43.67}{596}$ to four significant figures.

$$\begin{aligned} \log 85 &= 2.9294 \\ \log 43.67 &= \underline{1.6402} \\ \log(85 \times 43.67) &= 4.5696 \\ \log 596 &= \underline{2.7752} \\ \log N &= 1.7944 \\ N &= 62.29 \end{aligned}$$

In using logarithms, much time is saved and liability of error is decreased by making a so-called form for all the work before using the table at all.

Thus, in Example 1, the "form" is

$$\log 85 =$$
$$\log 43.67 =$$
$$\log (85)\ (43.67) =$$
$$\log 596 =$$
$$\log N =$$
$$N =$$

2. Make a form for evaluating $N = \dfrac{(478.9)^{\frac{1}{2}} \cdot \sqrt[3]{.0596}}{\sqrt[3]{834}}$.

$$\log 478.9 =$$
$$\log (478.9)^{\frac{1}{2}} =$$
$$\log .0596 =$$
$$\log \sqrt[3]{.0596} =$$
$$\log (478.9)^{\frac{1}{2}} \sqrt[3]{.0596} =$$
$$\log 834 =$$
$$\log \sqrt[3]{834} =$$
$$\log N =$$

EXERCISES AND PROBLEMS

Evaluate to four significant figures by logarithms:

1. 139×971.

2. 49.1×12.37.

3. $\dfrac{4000}{982.7}$.

4. 1.569^2.

5. $\sqrt{2}$.

6. $\sqrt[3]{10}$.

7. $\dfrac{1.06 \times 73.9}{19.14}$.

8. $\sqrt{.0537}$.

9. $\sqrt[3]{.02}$.

10. $\sqrt{\dfrac{4}{5}}$.

11. $\sqrt[3]{\dfrac{.2147}{-.0965}}$.

12. $538{,}947 \times \sqrt{.0003}$.

13. $\left(\dfrac{8999}{35.14}\right)^{\frac{5}{6}}$.

14. $\sqrt{\dfrac{258 \times 35^2}{86.75}}$.

15. $6442\sqrt[3]{\dfrac{-21.17}{3.142}}$.

16. The area of a circle of radius r is πr^2. Use $\frac{22}{7}$ for π.

(*a*) Find the area of a circle whose radius is 17.3 inches.

(*b*) Find the radius of a circle whose area is 456.7 square inches.

17. The volume of a sphere is $\frac{4}{3}\pi r^3$.

(*a*) Find the volume when $r = 12.7$ inches.

(*b*) Find r when the volume equals 148 cubic inches.

18. Find the weight in tons of a cylindrical marble column whose length l is 12.7 feet, and whose radius r is 2.9 feet, if marble is 2.7 times as heavy as water, and a cubic foot of water weighs 62.5 pounds. The formula for the volume of a cylinder is $V = \pi r^2 l$.

19. How many acres are there in a triangular field whose sides are 30 rods, 14.7 rods, and 23.5 rods?

If a, b, and c are the sides of a triangle, its area is $\sqrt{s(s-a)\,s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$.

20. The amount of P dollars at rate r for n years compounded annually is $A = P(1+r)^n$. What is the amount of \$50 for 10 years at 4%, compounded annually?

21. The side S, of a regular pentagon inscribed in a circle of radius r is $S = \frac{r}{2}\sqrt{10 - 2\sqrt{5}}$. Find S if $r = 1$.

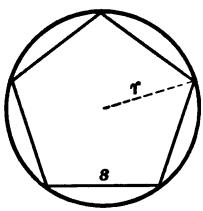

FIG. 50

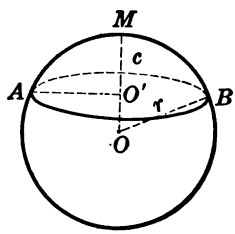

FIG. 51

22. The side of a regular pentagon is 10 inches. Find the radius of the circumscribed circle.

The portion of a sphere cut off by one plane is a special kind of spherical segment. (See Art. 45, Fig. 17.) The section made by the cutting plane is called the base of the segment. In the figure the circle ACB is the base; MO' is perpendicular to this circle; and MO' is the altitude of the segment. The volume V, of a spherical segment of altitude h, in a sphere of radius r, is $V = \pi h^2\left(r - \frac{h}{3}\right)$.

23. Find the volume of a spherical segment of altitude 2.67 inches in a sphere of radius 6.09 inches.

24. Show that the foregoing formula for the volume of a spherical segment can be deduced from the formula given in 18, Art. 45. (Note that here $b = 0$.)

25. An auger hole 1 inch in diameter is bored through the center of a croquet ball 4 inches in diameter. What is the volume of the wood removed?

26. Estimate the weight of a cork ball 6 feet in diameter and then calculate its weight in pounds. A cork whose volume is 1 cubic inch weighs 0.139 ounces.

27. First estimate and then calculate the increase in weight of the ball in Problem 26 if the radius is increased 1 inch.

28. The Dutch paid \$24 to the Indians in 1614 for Manhattan Island. What would this \$24 amount to in 1914 if it had been on interest at 5% compounded annually?

29. What would be the amount to-day of 1 cent which 1900 years ago was placed on interest at 6%, compounded annually? Find the radius in miles of a sphere of gold which has this value.

A cubic foot of gold weighs 1206 pounds avoirdupois; 1 such pound contains 7000 grains; and 23.22 grains of gold are worth a dollar.

Historical note on logarithms. Logarithms were invented by Baron John Napier (1550–1617) of Scotland.

The base used by Napier was not 10; but, soon after the publication of the invention, Briggs, a professor in London, visited Napier, and it seems probable that they both saw the usefulness of a table of logarithms to the base 10. Briggs devoted himself to the task of making such tables. For this reason, logarithms to the base 10 are often called Briggs' logarithms.

An early incident of the visit mentioned above is frequently quoted. It appears that Briggs and Napier were very anxious to have a visit, but that after meeting, about a quarter hour was spent, each beholding the other without saying a word. Finally Briggs spoke as follows: "My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this excellent help in astronomy, viz., the logarithms; but, my lord, being by you found out, I wonder why nobody found it out before, when now known it is so easy."

REVIEW EXERCISES AND PROBLEMS ON CHAPTERS IX–XII

1. Solve the following quadratic equations as indicated:

(*a*) $x^2 + 6x - 27 = 0$, by factoring.

(*b*) $y^2 + 10y - 3 = 0$, by completing the square.

(*c*) $z^2 - 2z = 11$, by formula. (WISCONSIN)

2. Solve $$(x-y)^2 + (x+y)^2 = 10,$$ $$(x-y)(x+y) = 3.$$

3. Simplify $\left[\frac{1}{a^{n-1}} \cdot a^{\frac{1}{n+1}}\right]^{\frac{1}{n}}$.

4. Determine for what values of a the roots of

$$4x^2 - 2(a-3)x + 1 = 0$$

are, (1) real and equal, (2) real and unequal, (3) imaginary. (MASS. INSTITUTE)

5. Reduce to a single number by means of logarithms

$$.000743 \times 23.56^{2.5} \div \sqrt[5]{4000.}$$ (YALE)

6. Compute $\frac{(57.1)^2 \times .9004}{721}$ by means of logarithms. (PENNSYLVANIA)

7. Solve $\frac{3x+2}{x+3} - \frac{2x+3}{x-3} + \frac{2}{9-x^2} = 0.$ (ILLINOIS)

8. Solve to three significant figures

$$x^2 - 1.6x - 0.23 = 0.$$ (HARVARD)

9. Simplify $\frac{(x^{-2}y^{-6})^{-\frac{1}{5}}}{(x^3y^{-2})} \cdot \frac{(y^3x^{-6})^{-1}}{(x^3y^{-3})}$ and express the result without using negative exponents. (CASE)

10. Solve the simultaneous equations by two methods:

$$x^2 + xy + y^2 = 39,$$ $$x + y = -2.$$ (WISCONSIN)

11. Simplify the following

(*a*) $2\sqrt{3} - \sqrt{12} + \sqrt[4]{9}.$

(*b*) $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{2}} \times \sqrt[3]{3}.$

(*c*) $\left(a^{2n+n} \cdot b^{\frac{1}{3n}}\right)^{2n}$. (SHEFFIELD)

12. Calculate $\log m$ and $\log n$, when $m = \dfrac{12.55}{42.27}$ and $n = (.0125)^{\frac{1}{3}}(423)^{\frac{1}{2}}$, given $\log 125 = 2.0967$, $\log 126 = 2.1004$, $\log 422 = 2.6253$, $\log 423 = 2.6263$. (NEBRASKA)

13. Find $\dfrac{\log 3}{\log 7} - \log \dfrac{3}{7}$.

14. How many digits in the number 2^{100}?

15. Solve $\dfrac{2}{1+3x} - \left(\dfrac{a}{b}\dfrac{(1+2x)}{(1+3x)} - \dfrac{b}{a}\dfrac{(3x-1)}{(2x+1)}\right) = 0$. (HARVARD)

16. (*a*) Find without solving, the sum and product of the roots of

$$-3x^2 + 12x + 11 = 0.$$

(*b*) Find an equation whose roots are $\dfrac{2}{3}$ and $\dfrac{3}{2}$.

(*c*) If $2 + 3\sqrt{-1}$ is one root of a quadratic equation, what is the other root? (MISSOURI)

17. Solve $x + \dfrac{1}{y} = 1,\ y + \dfrac{1}{x} = 4.$ (PRINCETON)

18. (*a*) For what value of c does the equation $6x^2 + 8x + c = 0$ have equal roots?

(*b*) Solve $16x^2 = 7x - 1$. (ILLINOIS)

19. Express without radicals in simplest form

$$(a)\ \sqrt{x\sqrt{x\sqrt{x}}}.\quad (b)\ \left(\sqrt[3]{\sqrt{27a^6}}\right)^{14}. \qquad \text{(NEBRASKA)}$$

20. Find a logarithmic expression for the amount of \$1.00 placed for 10 years at 5% compound interest compounded semiannually. (SHEFFIELD)

21. For what value of b will the simultaneous equations

$$16x^2 + 25y^2 = 400,$$

$$y = \frac{4}{5}x + b.$$

be satisfied by only one pair of values of x and y. (HARVARD)

22. Give the numerical value of

$$(a)\ 4^{\frac{3}{2}},\ (b)\ (-8)^{\frac{1}{3}},\ (c)\ 3^{-2},\ (d)\ 5^0. \qquad \text{(WISCONSIN)}$$

23. Solve the equation

$$2\log 6 - 5\log x = 3. \qquad \text{(CORNELL)}$$

24. The number of revolutions per minute of a water-wheel is given by the formula

$$n = 53.25\,\frac{H^{\frac{1}{2}}}{P^{\frac{1}{4}}}$$

where H is the fall of the water in feet, P the horse power. Calculate n for $H = 20$, $P = 50$; also for $H = 20$, $P = 100$.

25. A pays \$420 for the rent of a farm. He lets all but 4 acres to B, charging \$2.50 per acre more than he paid, and receives from B as much as he originally paid. How many acres in the farm? (SHEFFIELD)

26. A boat's crew, rowing at full speed, rows 2 miles down stream and back again in 35 minutes. Rowing at half full speed it takes 80 minutes to go over the same course. Find the rate of their rowing and the rate of the stream.

27. Perform the indicated operations:

(a) $(r^{-3}a^6) \cdot (r^2 x^{\frac{1}{2}})$.

(b) $t^{2.23} \div t^{1.03}$.

(c) $10^{9.23} \div 10^{1.5}$. (WISCONSIN)

28. Find the base of a system of logarithms in which $\log \frac{1}{2} = \bar{1}.8000$.

29. Simplify $\left[(a^{p+q})^{p-q} \cdot (a^{q^2})^{\frac{1}{q}}\right]^{\frac{1}{p^2}}$. (PRINCETON)

30. A man bought a flock of sheep for one hundred dollars. If he had bought five less for the same sum, they would have cost one dollar more per head. How many did he buy? (CASE)

31. A park is 120 rods long and 80 rods wide. It is decided to double the area of the park, still keeping it rectangular by adding strips of equal width to one end and one side. Find the width of the strip. (MISSOURI)

32. Given the equation

$$x^2 + bx - 18 = 0.$$

If one root is -9, what is the other, and what is the value of b?

(CALIFORNIA)

33. Express the solution of $4^{2x-1} = .5^{x-1}$ by using logarithms.

(SHEFFIELD)

34. Both the numerator and the denominator of a fraction are increased by their squares. The resulting fraction reduces to $\frac{1}{2}$. The sum of the numerator and denominator is 5. What is the fraction? (ILLINOIS)

35. Find the tenth root of one tenth.

36. Find the value of y from the equation

$$8^{\frac{1}{3}} + y^{-\frac{1}{2}} = \frac{7}{3 - \sqrt{2}}.$$ (HARVARD)

37. The plate of a mirror is 18 by 12 inches, and it is to be framed with a frame of uniform width whose area is to be equal to that of the glass. Find the width of the frame. (CASE)

38. The joint stock of two partners was \$2000. A's money was in trade 12 months and B's 15 months. A received \$1272, capital and profit. B received \$860 capital and profit. What was the capital of each?

39. What is the price of eggs when 2 less for 24 cents raises the price 2 cents a dozen? (YALE)

40. Which is the larger $2^{\frac{3}{2}}$ or $\sqrt{6}$? Prove your answer.

41. When a body is thrown vertically upwards, with an initial velocity of v_0 feet per second, the distance d from the point of departure after t seconds have passed, is $d = v_0 t - \frac{1}{2}gt^2$, where $g = 32.2$. If a body is thrown vertically upwards with an initial velocity of 64.4 feet per second find the time when $d = 48.3$ feet. Explain the two answers. (ILLINOIS)

42. What would be the answer to Problem 41 if, as is the case on the moon, the value of g were 5.4?

43. Given $$x^{\frac{1}{3}} + y^{\frac{1}{3}} = a^{\frac{1}{3}},$$

$$p = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}},$$

$$q = \tfrac{1}{3}\left(\frac{x}{y}\right)^{\frac{1}{3}}\left(\frac{y - px}{x^2}\right),$$

$$R = \frac{(1 + p^2)^{\frac{3}{2}}}{q}.$$

Express R in terms of x and y and in as simple a form as possible. (DARTMOUTH)

44. Which yields more, a sum left 10 years at 5%, or 5 years at 10%, interest compounded annually? What is the difference for \$1000?

45. A rectangular field is three times as long as it is broad. The shortening of the field by 3 rods and the broadening by 4 rods has the effect of increasing the area by one-half the original area. Find the original area in square rods. (PRINCETON)

46. Simplify $$\left[x^{\frac{1}{3}}\sqrt{\left(\frac{x^{\frac{1}{3}}}{\sqrt[3]{x}}\right)^5}\right]^{\frac{3}{16}}.$$ (ILLINOIS)

47. \$4400 is invested in two parts and at different rates of interest, so as to give the same income. If the first part were invested at the second rate, the income from this part would be \$100; but if the second part were invested at the first rate, its income would be \$144. What are the rates of interest?

48. A rectangle is 6 inches wide and 8 inches long. Keeping the width the same, how many inches must be added to the length to make the diagonal 1 foot long? (Correct to hundredths of inches.) (STANFORD)

49. Simplify

$$\frac{\sqrt{(1+x)(1+y)} - \sqrt{(1-x)(1-y)}}{\sqrt{(1+x)(1+y)} + \sqrt{(1-x)(1-y)}}.$$

50. At simple interest, \$300 amounted to \$405 at a certain rate and time. If the time had been one year longer and the rate 1 % less, it would have amounted to \$408. What were the time and the rate?

51. Find the value of x in the following equations:

(*a*) $\left(\frac{1}{a}\right)^{x+1} = a^3$; (*b*) $\log 5 + \log(3x - 2) = 3$. (DARTMOUTH)

CHAPTER XIII

EQUATIONS INVOLVING RADICALS

123. Solution of equations containing radicals. Certain equations in which the unknown is involved under the radical sign can be reduced to equations of the first or second degree. The following examples illustrate the method of solving some of the more simple but typical equations in which such reduction can be made.

EXAMPLES

1. Solve the equation $\sqrt{3x+1} = 5$.

SOLUTION: Squaring both members,

$$3x + 1 = 25.$$

Solving for x,

$$x = 8.$$

CHECK:

$$\sqrt{25} = 5.$$

It should be recalled that $\sqrt{25} = +5$, and does not equal ± 5; that is, when no sign precedes the radical the positive value of the root is to be taken. If both positive and negative roots are meant, we shall write both signs before the radical.

2. Solve $\sqrt{4x+5} + 2\sqrt{x-3} = 17$.

SOLUTION: Transposing,

$$\sqrt{4x+5} - 17 = -2\sqrt{x-3}.$$

Squaring,

$$4x + 5 - 34\sqrt{4x+5} + 289 = 4x - 12.$$

Transposing and simplifying,

$$\sqrt{4x+5} = 9.$$

Squaring,

$$4x + 5 = 81.$$

Solving for x,

$$x = 19.$$

CHECK:

$$\sqrt{81} + 2\sqrt{16} = 17,$$

or,

$$17 = 17.$$

3. Solve $\sqrt{x-2} - \sqrt{x+3} = 1$.

SOLUTION: Transposing $\sqrt{x+3}$ and squaring,

$$x - 2 = x + 3 + 2\sqrt{x+3} + 1.$$

This reduces to $\sqrt{x+3} = -3$.

Solving, $x = 6$.

But 6 is not a solution of the given equation. In fact, the given equation has no solution.

4. Solve $\sqrt{5x-4} = \sqrt{2x+1} + 1$.

SOLUTION: Squaring and transposing, $3x - 6 = 2\sqrt{2x+1}$.

Squaring and transposing again, $9x^2 - 44x + 32 = 0$.

Solving, $x = 4$, or $\frac{8}{9}$.

The value 4 satisfies the given equation, but $\frac{8}{9}$ does not. We notice, however, that $\frac{8}{9}$ does satisfy the equation if the signs of the radicals are taken negative.

5. Solve $2\sqrt{x+2} = \sqrt{x+1} - 2$.

SOLUTION: Squaring and transposing, $-4\sqrt{x+1} = 3x + 3$.

Squaring and transposing again, $9x^2 + 2x - 7 = 0$.

Solving, $x = -1$, or $\frac{7}{9}$.

Neither of these values is a root of the given equation.

The above examples illustrate that:

(1) Two expressions involving radicals may not be equal for any value of the unknown, as in Example 3.

(2) Extraneous roots may be introduced by squaring, as in Examples 4 and 5.

(3) Results must always be checked. There is no other way to determine whether or not a result is a root of the given equation. In Examples 3, 4, and 5, results are obtained which are not roots of the given equation.

Steps in the solution. In solving equations containing radicals it is usually convenient to proceed as follows:

(1) Isolate the radical; that is, place it by itself in one member of the equation. If more than one radical occurs, isolate the most complicated one.

(2) Square both members of the equation.

(3) If a radical remains, isolate it and square again.
(4) Solve the resulting equation.
(5) Check the results.

EXERCISES

1. $\sqrt{x+3} = 2.$
2. $\sqrt{3x-5} - 5 = 0.$
3. $\sqrt{3x-5} + 5 = 0.$
4. $\sqrt{x+5} - \sqrt{x} = 2.$
5. $\sqrt[3]{x+3} = 1.$

HINT: Cube each member.

6. $\sqrt[3]{x-4} + 7 = 9.$
7. $(2x-6)^{\frac{1}{2}} = 4.$
8. $10 - x = \sqrt{x^2-5}.$
9. $\sqrt{x+15} = \sqrt{x+1} + 2.$
10. $\sqrt{9x+6} - 3\sqrt{x-11} - 5 = 0.$
11. $x - 4 - \sqrt{2x-5} = 0.$
12. $\sqrt{10+x} - \sqrt{10-x} = 2.$
13. $\sqrt{2x+1} - x = \frac{1}{2}.$
14. $2\sqrt{x-3} + 3\sqrt{x-2} = \sqrt{13x-30}.$
15. $\sqrt{x+1} = \sqrt{9-x} - \sqrt{x+4}.$
16. $x + 5\sqrt{x} = -6.$
17. $\sqrt{3x+1} = \sqrt{5x-4} - \sqrt{x-7}.$
18. $\sqrt{x-7} - \sqrt{5x-4} = \sqrt{3x+1}.$
19. $\dfrac{\sqrt{x+7}}{\sqrt{x-5}} = \dfrac{1}{2}.$
20. $\dfrac{\sqrt{x}+\sqrt{2}}{\sqrt{x}+2} = 1.$
21. $\dfrac{\sqrt{x^2-2x+9}}{\sqrt{3x^2-x+7}} = \dfrac{2}{3}.$
22. $\sqrt{\dfrac{x-6}{x+12}} - \sqrt{\dfrac{2x-14}{3x+4}} = 0.$

23. $m\sqrt{x} - p = n\sqrt{x} - q.$

24. $\sqrt{a+x} + \sqrt{a-x} = \sqrt{2a}.$

25. $a + \sqrt{a^2 - x^2} = x.$

26. $\sqrt{m-x} + \sqrt{x-n} = \sqrt{m-n}.$

27. $\sqrt{x} + \dfrac{\sqrt{b} - \sqrt{a}}{\sqrt{b}} = \dfrac{1}{\sqrt{x}} + \dfrac{\sqrt{a} - \sqrt{b}}{\sqrt{a}}.$

28. $\dfrac{\sqrt{2x-3} + \sqrt{7x-6}}{\sqrt{2x-3} - \sqrt{7x-6}} = -3.$

29. $\sqrt{\dfrac{x-m}{x-n}} - \sqrt{\dfrac{x-n}{x-m}} = \dfrac{15}{4}.$

30. $\sqrt{x+a} - \sqrt{5x - 3a - 4b} = \dfrac{2b}{\sqrt{x+a}}.$

31. Solve $V = \dfrac{4}{3}\pi r^3$ for r.

32. Solve $A = \dfrac{\sqrt{3}s^2}{4}$ for s.

33. Solve $s = \dfrac{mgl}{\pi r^2 k}$ for r.

34. Solve $d = .02758\sqrt{D \cdot l \cdot \sqrt{P}}$ for P.

35. Solve $V = \dfrac{h}{3}(B + b + \sqrt{Bb})$ for B.

36. Solve $A = P(1+r)^t$ for r.

CHAPTER XIV

EQUATIONS IN QUADRATIC FORM

124. Quadratic form. Any equation is in **quadratic form** with respect to an expression containing the unknown, if we may replace such expression by a new letter, and obtain a quadratic in that new letter.

Thus, in the equation,

$$x - \sqrt{x-3} - 5 = 0,$$

which may be written

$$x - 3 - \sqrt{x-3} - 2 = 0,$$

if we let $z = \sqrt{x-3}$, we have $z^2 - z - 2 = 0$.

Similarly, in $2x^{-2} + x^{-\frac{1}{2}} + 1 = 0$,

if we let $z = x^{-\frac{1}{2}}$, we have $2z^2 + z + 1 = 0$.

Again, in $ax^{2n} + bx^n + c = 0$,

if we let $z = x^n$, we have $az^2 + bz + c = 0$.

EXERCISES

Solve the following equations and check results by substitution:

1. $x - 3\sqrt{x} + 2 = 0.$ (1)

SOLUTION: Let $\sqrt{x} = z.$

Then $x = z^2.$

Substituting in (1), $z^2 - 3z + 2 = 0.$

Factoring, $(z-1)(z-2) = 0.$

The roots are $z = 1,\ z = 2.$

Thus, $\sqrt{x} = 1,\ \sqrt{x} = 2,$

or $x = 1,\ x = 4.$

CHECK: $1 - 3 \cdot 1 + 2 = 0;\ 4 - 3 \cdot 2 + 2 = 0.$

2. $x^{-4} - 11x^{-2} + 30 = 0.$ (1)

SOLUTION: Let $x^{-2} = z.$ (2)

Then from (1) and (2), $z^2 - 11z + 30 = 0.$

Factoring, $(z - 5)(z - 6) = 0.$

Then $z = 5, z = 6.$

Hence, $\frac{1}{x} = \pm\sqrt{5}, \frac{1}{x} = \pm\sqrt{6}.$ (3)

and $x = \pm\frac{1}{\sqrt{5}} = \pm\frac{\sqrt{5}}{5}, x = \pm\frac{1}{\sqrt{6}} = \pm\frac{\sqrt{6}}{6}.$

The four numbers $+\frac{\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}, +\frac{\sqrt{6}}{6}, -\frac{\sqrt{6}}{6}$ are solutions of (1). The student should check by substitution in (1).

3. $x^4 + x^2 - 20 = 0.$ (1)

SOLUTION: Let $x^2 = z.$ (2)

Then from (1) and (2), $z^2 + z - 20 = 0,$ (3)

$(z - 4)(z + 5) = 0.$ (4)

Then $z = 4, z = -5.$ (5)

Hence, $x = \pm 2, x = \pm i\sqrt{5}.$ (6)

Note that two of the four roots are imaginary. The student should check by substitution in (1).

4. $x^{2n} - x^n - 56 = 0.$ (1)

SOLUTION: Let $x^n = z.$ (2)

Then from (1) and (2), $z^2 - z - 56 = 0.$

Solving by the formula, $z = \frac{1 \pm \sqrt{1 + 224}}{2} = 8 \text{ or } -7.$

Hence, $x = \sqrt[n]{8} \text{ or } \sqrt[n]{-7}.$

CHECK: $(\sqrt[n]{8})^{2n} - (\sqrt[n]{8})^n - 56 = 8^2 - 8 - 56 = 0.$

$(\sqrt[n]{-7})^{2n} - (\sqrt[n]{-7})^n - 56 = (-7)^2 + 7 - 56 = 0.$

5. $x^4 - 13x^2 + 36 = 0.$

6. $x^{-2} + 3x^{-1} + 2 = 0.$

7. $x^{\frac{1}{2}} - x^{\frac{1}{4}} = 6.$

8. $2 - 6x^{-2} - x^{-1} = 0.$

9. $\sqrt{x + 10} + \sqrt[4]{x + 10} = 2.$

10. $x + \sqrt{x + 6} = 14.$

11. $\sqrt{x} = 8 - \dfrac{15}{\sqrt{x}}$.

12. $\left(x + \dfrac{1}{x}\right)^2 + 4x + \dfrac{4}{x} = 12$.

13. $x^{-3} - 9x^{-\frac{3}{2}} + 8 = 0$.

14. $(x^2 + 5x)^2 + x(x + 5) = 42$.

15. $(4x^2 - 3)^2 + (8x^2 - 6)^2 = 80$.

16. $\sqrt{2x+1} - x = \frac{1}{2}$.

17. $3\sqrt[3]{x^2} + 6\sqrt[3]{x} - 4 = 0$.

18. $x^{-5} + 31x^{-\frac{5}{2}} - 32 = 0$.

19. $x^2 + 3x + 3\sqrt{x^2 + 3x + 2} + 2 = 0$.

20. $2x^2 - 4x + 3\sqrt{x^2 - 2x + 6} = 15$.

21. $x^4 + 2x^3 - x^2 - 2x - 3 = 0$.

HINT: Write the equation in the form $x^4 + 2x^3 + x^2 - 2(x^2 + x) - 3 = 0$ and make $x^2 + x = z$.

22. $x^4 - 8x^3 + 23x^2 - 28x - 8 = 0$.

23. $6x^{2n} - 7x^n - 20 = 0$.

24. $x^4 - 7a^2x^2 - 8a^4 = 0$.

25. $p^4 - 10p^2 - 11 = 0$.

26. $3(p^4 - 21) = 20p^2$.

27. $k^2(k^2 + 1) = 90$.

28. $16x^2(1 - x^2) = 3$.

Solve the following equations for $\log x$:

29. $(\log x)^2 - 5 \log x + 4 = 0$.

30. $3(\log x)^2 - 11 \log x - 4 = 0$.

Solve the following equations for 10^x.

31. $10^{2x} - 7.10^x + 6 = 0$.

32. $10^{2x} - 11.10^x + 10 = 0$.

33. If the square of a certain number is multiplied by this square increased by 6, the result is 40. Find the number. Are there any integers that satisfy this condition? Any imaginary numbers? Check results by substitution.

34. If the square of a certain number is multiplied by this square decreased by 4, the product is 192. Find numbers that satisfy this condition, and check results by substitution. Are the results integral? Fractional? Real? Imaginary?

125. Simultaneous quadratic form. Any system of equations is in **simultaneous quadratic form** with respect to expressions containing the unknowns, if we may replace such expressions by new letters, and obtain a system of simultaneous quadratics in such new letters.

Thus, in the system,

$$x^{-2} - y^{-2} = 6,$$
$$x^{-1} - y^{-1} = 2,$$

if we let $z = x^{-1}$, $w = y^{-1}$, we have

$$z^2 - w^2 = 6,$$
$$z - w = 2.$$

EXERCISES

Solve the following equations and check by substitution:

1. $x^{-2} - y^{-2} = 6,$ (1)
$x^{-1} - y^{-1} = 2.$

SOLUTION: Let $x^{-1} = z$, $y^{-1} = w$.

Then system (1) becomes $z^2 - w^2 = 6,$ (2)
$z - w = 2.$ (3)

Solve for z in (3) and substitute in (1),

$$w^2 + 4w + 4 - w^2 = 6, \quad (4)$$
$$4w = 2, \quad (5)$$
$$w = \tfrac{1}{2}. \quad (6)$$

From (6) and (3), $z = \tfrac{5}{2}$.

Hence, $x = \tfrac{2}{5}$, $y = 2$.

CHECK: $(\tfrac{2}{5})^{-2} - (2)^{-2} = (\tfrac{5}{2})^2 - (\tfrac{1}{2})^2 = 6,$
$(\tfrac{2}{5})^{-1} - (2)^{-1} = \tfrac{5}{2} - \tfrac{1}{2} = 2.$

2. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 13,$
$\dfrac{1}{xy} = 6.$

3. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = 10,$
$\dfrac{1}{xy} = 3.$

4. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 12,$
$\frac{1}{x} - \frac{1}{y} = 2.$

5. $\frac{4}{x^2} + \frac{9}{y^2} - \frac{13}{xy} = 19,$
$\frac{1}{xy} - \frac{1}{y^2} + 3 = 0.$

6. $2x + \frac{1}{y} = 6,$
$xy = 1.$

7. $x + y = 13,$
$\sqrt{x} - \sqrt{y} = 1.$

8. $x + \sqrt{y} = 7,$
$\sqrt{y} - \sqrt{x} = 1.$

9. $\left(x + \frac{1}{x}\right)^2 - \left(y + \frac{1}{y}\right)^2 = 4,$
$x + y + \frac{1}{x} + \frac{1}{y} = \frac{9}{2}.$

EXERCISES AND PROBLEMS

Solve the following equations for x:

1. $4x^4 - 5x^2 + 1 = 0.$
2. $3x^4 - 7x^2 - 6 = 0.$
3. $x^4 - 13x^2 + 40 = 0.$
4. $(x^2 - 1)^2 - 3(x^2 - 1) + 2 = 0.$
5. $x^2 - \frac{3}{x^2} = 2.$
6. $\frac{x - 4}{2 - x} = \frac{x}{12}.$
7. $\frac{x - 5}{x - 9} = x - 5.$
8. $x^{\frac{1}{2}} - 11x^{\frac{1}{4}} + 24 = 0.$
9. $10x^4 - 4x^2 - 32 = 0.$
10. $(3x + 4)^{\frac{1}{2}} + 5(3x + 4)^{-\frac{1}{2}} - 6 = 0.$
11. $x^4 - 4(a + b)x^2 + 4(a + b)^2 = 0.$
12. $x^4 - 4(a + b)x^2 + 16(a + b)^2 = 0.$
13. $35x^{-2} - 12x^{-1} + 1 = 0.$
14. $(x^2 - 5x + 6)^2 - 3(x^2 - 5x + 6) + 2 = 0.$
15. $(x^2 - 8)(x^2 - 3) = 374.$
16. $(4x^2 - 5x + 2)^2 = 12x^2 - 15x + 4.$
17. $x - 15\sqrt{x} + 56 = 0.$
18. $\frac{a^4}{x^2} - 4a^2 + 3x^2 = 0.$

Solve the following equations for x and y:

19. $\frac{1}{x}+\frac{1}{y}=\frac{3}{2}$,
$\frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{4}$.

20. $\frac{1}{x}+\frac{1}{y}=4$,
$\frac{1}{x^2}+\frac{1}{y^2}=\frac{17}{2}$.

21. $x-y=2$,
$\sqrt{x}+\sqrt{y}=2$.

22. $\frac{1}{x}+\frac{1}{y}=\frac{5}{3}$,
$xy=4$.

23. $(x-2y)^2-x+2y=6$,
$3x-5y=11$.

24. $\frac{x}{a}+\frac{y}{b}=3$,
$\frac{4ab}{x}-\frac{b^2}{y}=b$.

25. $x^{-2}+3y^2=7$,
$x^{-1}+4y^2=6$.

26. $(x-4)^2-(y-6)^2=0$,
$(x-4)^2+y=8$.

27. George Washington was born A.D. 1732. In the year A.D. x, he was $\sqrt{x}-10$ years old. Find the year A.D. x.

28. Find a number whose square multiplied by its square plus four gives 1440.

29. Find two numbers the sum of whose reciprocals is 5, and the sum of the squares of whose reciprocals is 13.

30. Find two numbers whose difference is 36, and the difference of whose square roots is 2.

31. The difference of the reciprocals of two numbers is 5, and the product of the numbers is .05. Find the numbers.

CHAPTER XV

PROGRESSIONS

126. Arithmetical progressions. An **arithmetical progression** is a succession of numbers each term of which, after the first, differs from the next preceding term by a fixed number called the **common difference.**

Thus, $2, 4, 6, 8, \cdots$

is an arithmetical progression with the common difference 2. In the arithmetical progression

$$10, 8, 6, 4, 2, \cdots$$

the common difference is -2.

127. Elements of an arithmetical progression. Let a represent the first term, d the common difference, n the number of terms considered, l the nth, or last, term, and s the sum of the terms. The five numbers $a, d, n, l,$ and s are called the **elements** of the arithmetical progression.

128. Relations among the elements. Since a is the first term, we have, by definition of an arithmetical progression,

$$\begin{aligned} a + d &= \text{second term}, \\ a + 2d &= \text{third term}, \\ a + 3d &= \text{fourth term}, \\ &\cdots\cdots\cdots \\ a + (n-1)d &= n\text{th term}. \end{aligned}$$

That is,
$$l = a + (n-1)d. \tag{1}$$

The sum of an arithmetical progression may be written in each of the following forms:

$$s = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l,$$
$$s = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$$

By addition

$$2s = (a + l) + (a + l) + (a + l) + \cdots + (a + l)$$
$$= n(a + l).$$

Therefore, $$s = \frac{n}{2}(a + l). \qquad (2)$$

Whenever any three of the five elements are given, equations (1) and (2) make it possible to find the remaining two elements.

129. Arithmetical means. The first and last terms of an arithmetical progression are called the **extremes,** while the remaining terms are called the **arithmetical means.** To insert a given number of arithmetical means between two numbers it is necessary only to determine d by the use of equation (1) and to write down the terms by the repeated addition of d.

Example. Insert 6 arithmetical means between 3 and 8.

SOLUTION: We have to find d, when $a = 3$, $l = 8$, and $n = 6 + 2 = 8$.

Since $$l = a + (n - 1)d,$$

we have $$8 = 3 + 7d, \text{ or } d = \frac{5}{7}.$$

Hence, the 6 arithmetical means between 3 and 8 are

$$\frac{26}{7}, \frac{31}{7}, \frac{36}{7}, \frac{41}{7}, \frac{46}{7}, \frac{51}{7}.$$

EXERCISES

Find l and s for the following five exercises:

1. $2 + 11 + 20 + \cdots$ to 10 terms.

SOLUTION: $$l = a + (n - 1)d.$$

Here, $$a = 2, d = 9, n = 10.$$
$$l = 2 + 9 \cdot 9 = 83.$$
$$s = \frac{n}{2}(a + 1),$$
$$= 5(2 + 83) = 425.$$

2. $-4-8-12-\cdots$ to 19 terms.

3. $\frac{5}{3}+\frac{7}{4}+\frac{11}{6}+\cdots$ to 21 terms.

4. $-\frac{7}{3}-\frac{40}{21}-\frac{31}{21}-\cdots$ to 17 terms.

5. $2+\frac{5}{7}-\frac{4}{7}-\frac{13}{7}-\cdots$ to 25 terms.

6. Given $d = 3$, $n = 25$, $l = 74$; find a and s.

7. Given $l = -\frac{1}{2}$, $n = 14$, $d = \frac{1}{2}$; find a and s.

8. Given $a = 3$, $l = 108$, $s = 1221$; find n and d.

9. Given $a = 2$, $l = 9$, $d = \frac{1}{3}$; find n and s.

10. Given $a = -4$, $l = -64$, $n = 21$; find d and s.

11. Given $d = \frac{1}{2}$, $l = \frac{74}{7}$, $s = \frac{1633}{14}$; find a and n.

12. Given $l = 41$, $n = 41$, $s = 861$; find a and d.

13. Given $a = -2$, $d = \frac{2}{19}$, $s = 0$; find n and l.

14. Given $a = 7$, $n = 7$, $s = 7$; find d and l.

15. Given $d = 10$, $n = 10$, $s = 10$; find a and l.

16. Insert 5 arithmetical means between 3 and 21.

17. Insert 8 arithmetical means between 9 and 81.

18. Find the arithmetical mean between $8\frac{1}{3}$ and $16\frac{2}{3}$.

19. Insert 11 arithmetical means between 2 and 20.

130. Geometrical progressions. A succession of terms in which the same quotient is obtained by dividing any term by the preceding term is called a **geometrical progression.** This quotient is called the **ratio.**

Thus,

$$3, 6, 12, 24, \cdots$$

is a geometrical progression with a ratio 2.

131. Elements of a geometrical progression. The elements are the same as those for an arithmetical progression with one exception. Instead of the common difference of an arithmetical progression, we have here a ratio represented by r.

132. Relations among the elements. If a represents the first term, then

$$ar = \text{second term},$$
$$ar^2 = \text{third term},$$
$$ar^3 = \text{fourth term},$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$ar^{n-1} = n\text{th term}.$$

That is, if l represents the nth term, we have

$$l = ar^{n-1}. \tag{1}$$

By definition,

$$s = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}, \tag{2}$$

Then,
$$sr = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{3}$$

Subtracting members of (2) from members of (3), we have

$$sr - s = ar^n - a.$$

Hence,
$$s = \frac{ar^n - a}{r - 1} = a\frac{(1 - r^n)}{1 - r}. \tag{4}$$

Since $l = ar^{n-1}$, (4) may be written in the form

$$s = \frac{rl - a}{r - 1}. \tag{5}$$

Here, as in an arithmetical progression, whenever any three of the five elements are given, relations (1) and (5) make it possible to find the other two.

133. Geometrical means. The first and last terms of a geometrical progression are called the **extremes,** while the remaining terms are called the **geometrical means.** To insert n geometrical means between two given numbers is to find a

geometrical progression of $n + 2$ terms having the two given numbers for extremes.

Example. Insert four geometrical means between 2 and 64.

SOLUTION: In this case

$$a = 2,$$

$$l = 64,$$

and $$n = 6.$$

Then, by (1), Art. 132 $$64 = 2r^5,$$

$$32 = r^5,$$

and $$r = 2.$$

Hence, the four geometrical means are 4, 8, 16, 32.

EXERCISES

1. Given $a = 5$, $r = 3$, $n = 10$; find l and s.

2. Given $a = -3$, $r = 2$, $n = 8$; find l and s.

3. Given $a = \frac{1}{2}$, $r = \frac{1}{2}$, $n = 10$; find l and s.

4. Given $a = 5$, $r = -2$, $n = 9$; find l and s.

5. Given $s = 50$, $a = 5$, $n = 10$; find r and l.

6. The third term of a geometrical progression is 3, and the sixth term is 81. What is the tenth term?

7. What is the fifth term of a geometrical progression whose first term is 2 and third term 4?

8. What is the sum of the first five terms of a geometrical progression whose first term is 2 and third term 8?

9. The first term of a geometrical progression is 4, and the last term 256. If there are four terms in the geometrical progression, find the common ratio and the sum of the series.

10. Insert one geometrical mean between 6 and 150.

11. Insert two geometrical means between 2 and 250.

12. Insert three geometrical means between 12 and $\frac{3}{4}$.

13. What is the eighth term of the series $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{18}$, . . . ?

14. If each term of a geometrical progression is multiplied by the same number, show that the products form a geometrical progression.

134. Number of terms infinite. Consider the geometrical progression

$$\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16} \cdots.$$

It may at first thought appear that the sum of the first n terms could be made to exceed any finite number previously assigned by making n large enough. That this is not the case and that the sum can never exceed unity, will be seen from the following illustration. Conceive a particle moving in a straight line toward a point one unit distant in such a way as to describe $\frac{1}{2}$ the distance in the first second, $\frac{1}{2}$ the remaining distance in the second second, $\frac{1}{2}$ the remaining distance in the third second, and so on indefinitely. This is represented in Fig. 52.

The distance AB represents one unit of distance. In the first second the particle moves from A to P_1. In the second second it moves from P_1 to P_2, and so on. The total distance traversed by the particle in n seconds is given by the sum

$$\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \cdots \text{ to } n \text{ terms},$$

which sum cannot exceed nor equal 1, no matter how many terms we take, but can be made to differ from 1 by as small a positive number as we please by making the number of terms large enough. Thus, when $n = 10$, the sum is $\dfrac{1023}{1024}$ (Exercise 3, Art. 133). In this illustration, 1 is said to be the limiting value

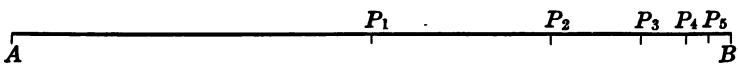

FIG. 52

of the sum of the first n terms of the progression. If s_n represents the sum of the first n terms, we write

$$\lim_{n=\infty} s_n = 1,$$

which reads, " the limit of s_n as n increases indefinitely is 1." The **sum** s of the infinite progression is defined as this limit.

For any geometrical progression in which the ratio is less than 1, the above argument can be repeated, and it can be

shown that there is a limiting value to the sum of the first n terms of such a progression. In Art. 132, we have shown that the sum,

$$a + ar + ar^2 + \cdots + ar^{n-1},$$

is given by $$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

We then write

$$\lim_{n = \infty} s_n = \lim_{n = \infty} \frac{a}{1 - r} - \lim_{n = \infty} \frac{ar^n}{1 - r}.$$

We shall consider only the case where $r < 1$. In this case, as n increases, the value of r^n becomes smaller and smaller. In fact, we may take n so large that $\frac{ar^n}{1 - r}$ is smaller in absolute value than any number that has been assigned and left fixed, or as we say, $\frac{ar^n}{1 - r}$ approaches the limit 0 as n becomes infinite.

Hence, the sum of the infinite geometrical progression

$$a,\ ar,\ ar^2, \cdots, \text{ when } r < 1,$$

is

$$s = \lim_{n = \infty} s_n = \frac{a}{1 - r}.$$

EXERCISES

Find the sum of the following infinite progressions:

1. $6, 2, \frac{2}{3}, \frac{2}{9}, \cdots$.

SOLUTION: Here $a = 6, r = \frac{1}{3}$.

$$s = \frac{a}{1 - r} = \frac{6}{1 - \frac{1}{3}} = 9.$$

2. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$.

3. $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \cdots$.

4. $32, 16, 8, 4, \cdots$.

5. $2, \frac{1}{4}, \frac{1}{32}, \cdots$.

6. $.5, .125, .03125, \cdots$.

7. $\sqrt{2}, 1, \frac{1}{\sqrt{2}}, \frac{1}{2}, \cdots$

8. $2, .25, .03125, \cdots$.

9. $.3, .03, .003, \cdots$.

135. Repeating decimals. Repeating decimals furnish good illustrations of infinite geometrical progressions. For example, $.66666 \cdots$ may be written as

$$.6 + .06 + .006 + .0006 + .00006 + \cdots$$

where $a = .6$ and $r = .1$. The limiting value of the sum of n terms as the number n increases indefinitely is $\frac{2}{3}$.

Again, $.4919191 \cdots$ may be written

$$.4 + .091 + .00091 + \cdots$$

where the terms after the first form a geometrical progression in which $a = .091$ and $r = .01$.

EXERCISES

Find the limiting value of each of the following repeating decimals:

1. $.7777 \cdots$.	**6.** $.83333 \cdots$.	**11.** $.404040 \cdots$.
2. $.5555 \cdots$.	**7.** $.727272 \cdots$.	**12.** $.242424 \cdots$.
3. $.16666 \cdots$.	**8.** $.505050 \cdots$.	**13.** $1.181818 \cdots$.
4. $.363636 \cdots$.	**9.** $.234234 \cdots$.	**14.** $2.272727 \cdots$.
5. $.909090 \cdots$.	**10.** $.609609 \cdots$.	**15.** $3.363636 \cdots$.

136. Harmonical progressions. Three or more numbers are said to form a **harmonical progression** if their reciprocals form an arithmetical progression. The term " harmonical " as here used comes from a property of musical sounds. If a set of strings of uniform tension whose lengths are proportional to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \cdots$ be sounded together, the effect is harmonious to the ear. The succession of numbers

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \cdots$$

is a harmonical progression since the reciprocals form the arithmetical progression

$$1, 2, 3, 4, 5, 6, \cdots$$

137. Harmonical means. To find n harmonical means between two numbers, find n arithmetical means (Art. 129) between the reciprocals of these numbers. The reciprocals of the arithmetical means are the **harmonical means.**

EXERCISES

1. Insert a harmonical mean between $\frac{1}{2}$ and $\frac{1}{8}$.
2. Insert two harmonical means between 1 and 4.
3. What is the harmonical mean between x and y?
4. Show that $\frac{3}{2}$, 2, 3 are in harmonical progression and continue the series two terms in each direction.

MISCELLANEOUS EXERCISES AND PROBLEMS

1. Find the seventh term of the progression $18, -6, 2, \cdots$.
2. Find the fifth term of the progression $a, a + d, a + 2d, \cdots$.
3. Find the tenth term of the progression $\frac{1}{n}, \frac{1}{n+2}, \frac{1}{n+4}, \cdots$.
4. What is the arithmetical mean between a and b?
5. Insert 5 arithmetical means between -4 and 20.
6. Insert 3 geometrical means between 5 and 405.
7. What is the geometrical mean between a and b?
8. The fourth term of a geometrical progression is 108, the sixth is 972. Find the ninth term.
9. The sum of three numbers in arithmetical progression is 15; if 1, 4, 19 are added to the numbers, the results are in geometrical progression. Find the numbers.
10. Show that the sum of the first n odd numbers 1, 3, 5, 7, $\cdots$ is n^2.

PROBLEMS PERTAINING TO MOTION

11. If a body falls 16 feet the first second, 48 the next, 80 the next and so on, how far does it fall in the twelfth second? How far has it fallen at the end of 12 seconds?

12. If a body falls 16 feet the first second, 48 the next, 80 the next and so on, how far does it fall in the tth second? How far has it fallen at the end of the tth second?

13. A ball rolling down an incline of 30° goes 8 feet the first second, and in each second thereafter 16 feet more than in the preceding second. How far will it roll in 10 seconds?

14. A marble rolls down an inclined plane, passing over distances 3 feet, 9 feet, 15 feet, in successive seconds. How long will it take it to pass over 108 feet?

15. Assume that a ball falls 16 feet the first second, 48 the next, 80 the next, and so on. A baseball was dropped from the top of Washington Monument, 550 feet high, and caught by an American League catcher. About how fast was the ball falling when caught?

16. In a potato race there are placed 20 potatoes, at the distances 5 feet, 8 feet, 11 feet, and so on, from a bag. A man starting from the bag is required to pick up the potatoes and carry them back to the bag one by one. What is the total distance that he goes in thus collecting the potatoes?

17. A wheel of perimeter 6 feet, free to rotate on an axis, is started to rotate 20 revolutions per second, and then is to be retarded by friction. If it makes 95% as many revolutions each second thereafter as it did the preceding second, how far will a point on the rim have moved when the wheel comes to a standstill?

HINT: These distances are assumed to form an infinite progression.

18. If a particle moves in a straight line from a given position, with such a speed that during any given second it moves 60% as far as it did during the preceding second, and if it moved 25 feet during the first second, what is the limit of the distance the particle will move?

19. If a ball should fall 9 feet and bound back 6 feet, then fall 6 feet and bound back 4 feet and so on indefinitely, what is the limit of the distance through which the ball would move?

20. *A* travels uniformly 25 miles a day; *B* starts at the same time, and travels 5 miles the first day, 10 miles the second, 15 miles the third, and so on. In how many days will *B* overtake *A*?

PROBLEMS PERTAINING TO BUSINESS

21. Find the present value of twenty annual payments of \$1000 made at end of each of the next twenty years, if money is worth 5% compounded annually.

HINT: The present value of \$1000 due in n years is $\frac{\$1000}{(1.05)^n}$.

22. Find the present value of \$125 per year paid at the end of each year for forty years if interest is at 4% compounded annually.

23. I owe a debt of \$4675, to be paid in instalments, the first payment to be \$800, the second \$725, and decreasing by a common difference, until the last payment which is \$50. Find the number of instalments.

24. A gentleman being importuned to sell a fine horse, said he would sell him on the condition of receiving 1 cent for the first nail in four shoes, 2 cents for the second, and so on, doubling the price of every nail; the number of nails in each shoe being 8, how much would he receive for his horse?

CHAPTER XVI

THE BINOMIAL THEOREM

138. Factorial. The symbols

$$r! \text{ and } \lfloor r$$

are read " factorial r," and are used to indicate the product $1 \cdot 2 \cdot 3 \cdot \ldots r$. The symbol $r!$ is easier to set up in type than the symbol $\lfloor r$, and is used more in print than $\lfloor r$.

Thus, $3! = 1 \cdot 2 \cdot 3 = 6$; $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$.

139. Powers of a binomial. By actual multiplication,

$$(a + b)^2 = a^2 + 2ab + b^2, \tag{1}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \tag{2}$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4. \tag{3}$$

Exercise. Expand $(a + b)^5$ by actual multiplication of (3) by $a + b$.

140. The expansion of $(a + b)^n$. The expressions (1), (2), and (3) of Art. 139 are special cases of the general formula

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots. \tag{4}$$

This is called the **binomial formula** or **binomial theorem.**

The following properties of this formula should be noted:

(1) The first term is a^n.

(2) The second term is $a^{n-1}b$.

(3) The exponents of a decrease by unity from term to term while the exponents of b increase by unity.

(4) If in any term the coefficient be multiplied by the exponent of a and divided by the exponent of b increased by 1, the result is the coefficient of the next term.

(5) The factorial number in the denominator of the coefficient of any term is equal to the exponent of b in that term.

(6) The numerator of the coefficient of any term is the product of factors beginning with n, decreasing successively by 1, and ending with $n - r + 1$, where r is the exponent of b.

(7) When n is any positive integer, the expression contains $n + 1$ terms of which the last is b^n.

The binomial theorem holds for any positive integral value of n, and, under certain restrictions on a and b, for negative and fractional values of n. (See Art. 142.)

The theorem is proved in more advanced algebra, but here it is merely stated without proof.

EXERCISES

1. Expand $(a - b)^n$.

SOLUTION: Put $-b$ for b in (4), Art. 140. The even powers of $-b$ are positive and the odd powers are negative. We may therefore write

$$(a - b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2$$

$$- \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + (-b)^n.$$

The last term is either $-b^n$ or $+b^n$ depending upon whether n is odd or even.

2. Expand $(3y + 5x)^5$.

SOLUTION: Substitute $3y$ for a, $5x$ for b, and 5 for n in (4), Art. 140.

$$(3y + 5x)^5 = (3y)^5 + 5(3y)^4 5x + \frac{5 \cdot 4}{1 \cdot 2}(3y)^3 (5x)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}(3y)^2 (5x)^3$$

$$+ \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}(3y)(5x)^4 + (5x)^5 = 243y^5 + 2025y^4x + 6750y^3x^2 + 11250y^2x^3$$

$$+ 9375yx^4 + 3125x^5.$$

3. Expand $\left(\frac{c}{3} - \frac{2b}{x}\right)^6$.

SOLUTION: Substitute $\frac{c}{3}$ for a, $-\frac{2b}{x}$ for b, and 6 for n in (4) Art. 140.

Then

$$\left(\frac{c}{3} - \frac{2b}{x}\right)^6 = \left(\frac{c}{3}\right)^6 + 6\left(\frac{c}{3}\right)^5\left(-\frac{2b}{x}\right) + 15\left(\frac{c}{3}\right)^4\left(-\frac{2b}{x}\right)^2 + 20\left(\frac{c}{3}\right)^3\left(-\frac{2b}{x}\right)^3$$

$$+ 15\left(\frac{c}{3}\right)^2\left(-\frac{2b}{x}\right)^4 + 6\left(\frac{c}{3}\right)\left(-\frac{2b}{x}\right)^5 + \left(-\frac{2b}{x}\right)^6 = \frac{c^6}{729} - \frac{4bc^5}{81x}$$

$$+ \frac{20b^2c^4}{27x^2} - \frac{160b^3c^3}{27x^3} + \frac{80b^4c^2}{3x^4} - \frac{64b^5c}{x^5} + \frac{64b^6}{x^6}.$$

Expand:

4. $(x - y)^8$.

5. $(1 + x^{-1})^{10}$.

6. $(x^2 - 2y)^6$.

7. $(x - x^{-1})^5$.

8. $(2x + \sqrt{x})^4$.

9. $\left(\frac{1}{x} + \frac{x}{2}\right)^5$.

10. $\left(\frac{x}{a} - \frac{a}{x}\right)^6$.

11. $\left(1 + \frac{3}{x^2}\right)^7$.

12. $(\sqrt{2} - \sqrt{3})^6$.

13. $\left(\sqrt{2} - \frac{1}{\sqrt{c}}\right)^4$.

14. $(x + x\sqrt{x})^4$.

15. $(x^{-2} - \sqrt{x})^4$.

16. $\left(\frac{b}{2} - bx\right)^6$.

17. $\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)^6$.

18. $\left(x + \frac{1}{x}\right)^6$.

19. $(x^{-1} + y^{\frac{1}{2}})^6$.

20. $(x + y + z)^3$.

HINT: Consider $x + y$ as representing one term of a binomial.

21. $\left(\frac{x^2}{2} + \frac{x}{3} + 1\right)^4$.

22. $\left(\sqrt{x} + 4 + \frac{1}{x}\right)^3$.

23. $(x^{\frac{1}{2}} + 2y^2)^4$.

Use the binomial formula to compute the following, correct to three significant figures:

24. $(1.1)^{20}$.

HINT: Write $(1.1)^{20} = (1 + .1)^{20}$ and expand to a few terms.

25. $(.99)^{20}$.

HINT: Write $(.99)^{20} = (1 - .01)^{20}$ and expand.

26. $(1.01)^{15}$. **27.** $(.999)^{10}$.

141. Formula for the rth term of the expansion of $(a + b)^n$. We have seen by Art. 140 that the third term is

$$\frac{n(n-1)}{2!}a^{n-2}b^2,$$

the fourth term is $\dfrac{n(n-1)(n-2)}{3!}a^{n-3}b^3$,

and hence that the rth term is

$$\frac{n(n-1)\ (n-2)\ \ldots\ (n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}.$$

EXERCISES

1. Find the sixth term of $\left(2x - \dfrac{y}{3}\right)^{10}$.

SOLUTION: Here $a = 2x$, $b = -3y$, $n = 10$, $r = 6$.

The sixth term is $\dfrac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}(2x)^5\left(-\dfrac{y}{3}\right)^5 = -\dfrac{896}{27}x^5y^5$.

2. In $(x - y)^{10}$, find the sixth term.

3. In $(x^2 - y^2)^{12}$, find the middle term.

4. In $(x^{\frac{1}{2}} - y^{\frac{1}{2}})^{14}$, find the middle term.

5. In $(3x - \frac{1}{3}a^2)^{15}$, find the sixth term.

6. In $\left(\dfrac{5}{x} + \dfrac{x}{5}\right)^{12}$, find the middle term.

7. In $(.5\sqrt{x} - .1\sqrt{y})^7$, find the fifth term.

142. Binomial formula, any exponent. The binomial formula $(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$.

holds when n is a negative or fractional number, provided the number b lies between $-a$ and a.

If n is not a positive integer, the formula for the development of $(a+b)^n$ contains an unlimited number of terms. If b is small compared to a, the first few terms of the expansion give a useful approximation for certain purposes. Thus, the first few terms in the binomial expansion of $(27+1)^{\frac{1}{3}}$ gives the approximate value of the cube root of 28. (See Example 3 below).

EXERCISES

1. Expand to four terms $(x+y)^{\frac{1}{3}}$.

SOLUTION: Substitute in the expansion for $(a+b)^n$. This gives

$$(x+y)^{\frac{1}{3}} = x^{\frac{1}{3}} + \frac{1}{3}x^{\frac{1}{3}-1}y + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}x^{\frac{1}{3}-2}y^2 + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}x^{\frac{1}{3}-3}y^3 + \cdots$$

$$= x^{\frac{1}{3}} + \frac{1}{3}x^{-\frac{2}{3}}y - \frac{1}{9}x^{-\frac{5}{3}}y^2 + \frac{5}{81}x^{-\frac{8}{3}}y^3 \cdots$$

2. Expand to three terms $(3x - \frac{1}{2}y)^{-2}$.

SOLUTION: By the formula for $(a+b)^n$,

$$\left(3x - \frac{1}{2}y\right)^{-2} = (3x)^{-2} - 2(3x)^{-3}\left(-\frac{1}{2}y\right) + \frac{-2(-3)}{2!}(3x)^{-4}\left(-\frac{1}{2}y\right)^2 + \cdots$$

$$= \frac{1}{9x^2} + \frac{y}{27x^3} + \frac{y^2}{108x^4} + \cdots$$

3. Find approximately $\sqrt[3]{28}$ by the binomial formula.

SOLUTION: Write

$$\sqrt[3]{28} = (27+1)^{\frac{1}{3}} = 27^{\frac{1}{3}} + \frac{1}{3}27^{-\frac{2}{3}} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}27^{-\frac{5}{3}} + \cdots$$

$$= 3 + \frac{1}{27} - \frac{1}{2187} +$$

$$= 3.0366 -.$$

Expand the following expressions to four terms:

4. $(1 - x)^{-3}$.

5. $(a - x)^{-4}$.

6. $(2a + 1)^{-5}$.

7. $(x - 2y)^{-6}$.

8. $(1 - x)^{-1}$.

9. $(1 + x)^{-\frac{1}{2}}$.

10. $(x^2y^{-2} + 1)^{\frac{1}{2}}$.

11. $(8a^3 + 1)^{\frac{1}{3}}$.

12. $(8x^{-3} - 1)^{\frac{1}{3}}$.

13. $(9x^2 + 2y^2)^{-\frac{1}{2}}$.

14. $\left(x - \frac{1}{x}\right)^{-1}$.

15. $\sqrt[3]{9} = (8 + 1)^{\frac{1}{3}}$.

Find the roots indicated, correct to three significant figures:

16. $\sqrt{10}$.

17. $(26)^{\frac{1}{2}}$.

18. $(79)^{\frac{1}{2}}$.

19. $\sqrt[3]{65}$.

20. $\sqrt[3]{730}$.

21. $(25)^{\frac{1}{2}}$.

REVIEW EXERCISES AND PROBLEMS ON CHAPTERS XIII-XVI

1. Find the arithmetic mean, the geometric mean, and the harmonic mean between a and b.

2. In a Christmas savings club there are three classes of depositors. One class deposits 1 cent the first week, 2 cents the second week, 3 cents the third week, and so on, increasing the weekly deposit 1 cent each week for 50 weeks; the second class deposits 2 cents the first week and increases the weekly deposit 2 cents each week for 50 weeks; the third class deposits 5 cents the first week, and increases the weekly deposit 5 cents each week for 50 weeks. Find how much is deposited by a member of each class in 50 weeks.

3. The population of a town is P at a certain time. Annually it loses $x\%$ by deaths and gains $y\%$ by births. What is the population at the end of n years?

4. What is the outer diameter of a spherical shell that is 1 inch thick and contains 50 cubic inches?

5. Solve for x (PRINCETON)

$$(a)\ x^2 + 2a^2 = 3ax.$$
$$(b)\ x^{\frac{1}{2}} + 3x^{-\frac{1}{2}} = 4.$$

6. What term in the development of $\left(a + \frac{1}{a}\right)^{2n}$ does not contain a. (SHEFFIELD)

7. Let a, b, c (Fig. 53) be three verticals erected so that a line from the foot of a to the top of c, or from the foot of c to the top of a, bisects the intermediate vertical b. Show that b is the harmonic mean between a and c; that is, show that

$$\frac{1}{2}\left(\frac{1}{a} + \frac{1}{c}\right) = \frac{1}{b}.$$

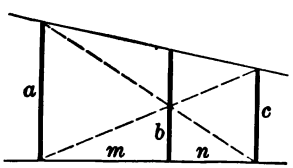

FIG. 53

8. In Fig 54 the lines $a, b, c, d, \cdots$ are drawn so that the angles marked 1, 2, 3, $\cdots$ are equal. Show that the lengths of $a, b, c, d, \cdots$ form a geometrical progression.

HINT: Show that $\frac{a}{b} = \frac{b}{c}$ by use of similar triangles, p. 99.

9. Solve $(x + 11)^{\frac{1}{2}} + x^{\frac{1}{2}} = 55x^{-\frac{1}{2}}$. (ILLINOIS)

10. Solve $\sqrt{x + 1} + x = 11$.

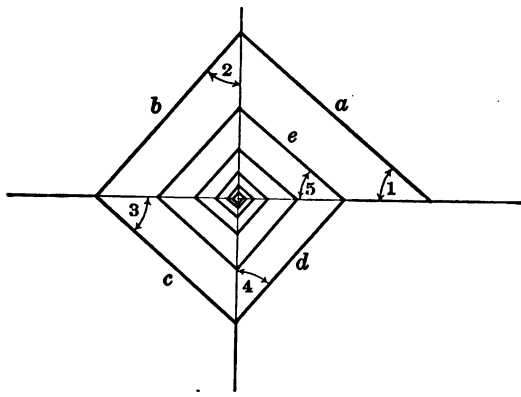

FIG. 54

11. Find to two places decimal by the binomial theorem the value of $(120)^{\frac{1}{4}}$. (CALIFORNIA)

12. A piece of land is in the form of a trapezoid, with parallel sides 20 rods and 30 rods, and altitude 20 rods. It is desired to divide this piece of land into two equal parts by a line parallel to the parallel sides. Find the distance between the shorter parallel side and this dividing line.

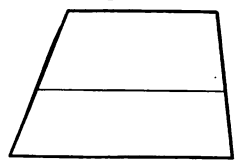

FIG. 55

13. Find the value of $\log_7 49 + \frac{1}{3}\log_2 64 + \log_6 216$.

14. Insert two arithmetical means, two geometrical means, and two harmonical means between 3 and 24.

15. Find the fifth term of $\left(x^4 + \frac{1}{5x^2}\right)^{10}$ and reduce to the simplest form. (DARTMOUTH)

16. Solve for x, $(\log x)^2 - 11 \log x + 10 = 0$.

17. The following quaint problem was found in an old Hindu manuscript: The square root of half the number of bees in a swarm have flown out upon a jessamine bush; $\frac{8}{9}$ of the whole swarm have remained behind; one female bee flies about a male that is buzzing within a lotus flower into which he was allured in the night by its sweet odor, but is now imprisoned in it. Tell me the number of bees in the swarm.

18. Solve for x and y

$$(x+2)^{\frac{1}{2}} - (2y-3)^{\frac{1}{2}} = 2,$$
$$x - 2y + 3 = 14.$$

19. Find the sum of n terms of the series,

$$1 + \frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \cdots.$$

(MASS. INSTITUTE)

20. A rectangle has the base x and altitude y, and a constant area 12. Construct the rectangles on coördinate paper when x has the values 24, 12, 8, 6, 4, 3, 2, 1, .5, .4, .3. In each case take the origin as one vertex and the coördinate axes as sides. After all the rectangles are constructed draw a curve through the vertices that are opposite the origin.

21. Gerbert (1003 A.D.), who was later Pope Sylvester, found the altitude of an equilateral triangle by multiplying the side by $\frac{6}{7}$; Heron (about 100 B.C.) used the factor $1\frac{3}{5}$; the factor $\frac{7}{4}$ is now used to make a rough approximation. Which value is nearest the correct one?

22. Continue the progression $\frac{1}{2}$, $\frac{1}{6}$, $-\frac{1}{8}$ two more terms and find the sum of the first 13 terms. (ILLINOIS)

23. Find the value to four decimal places of $\frac{1}{\sqrt{47}}$ by the binomial theorem. (MASS. INSTITUTE)

24. Insert an arithmetical mean between 9π and 4π, also a geometrical mean. How do they compare? (DARTMOUTH)

25. Solve for x $\quad 20(\log x)^2 - 23 \log x + 6 = 0.$

26. Solve for x $\quad 6(\log x)^2 + 17 \log x + 7 = 0.$

27. A number is 12 greater than $4\frac{1}{2}$ times its square root. Compute its value to two places of decimals. Verify the smaller value.

(COLLEGE ENTRANCE EXAMINATION BOARD)

28. (*a*) Of a geometric progression, the sum of the first and second terms is 16, the sum of the third and fourth terms is 36. Find the ratio, and the sum of the first 6 terms. (*b*) Write the seventh term of the expansion of $(1 + \sqrt{x})^9$ by the binomial theorem.

(COLLEGE ENTRANCE EXAMINATION BOARD)

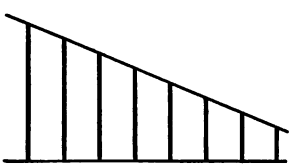

FIG. 56

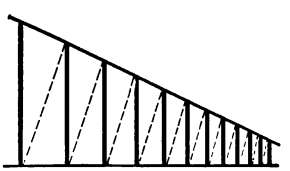

FIG. 57

29. Find the roots of $49x^4 - 70x^2 + 12 = 0$ to three decimal places.

30. The sum of an infinite geometric series is 4, and the first term is 6. Find the ratio and the sum of 4 terms. (YALE)

31. In Fig. 56, the vertical lines are equally spaced. Show that their lengths form an arithmetical progression.

32. In Fig. 57, the vertical lines are so spaced, that the lines drawn from the foot of

each to the top of the next are all parallel. Show that the lengths of the verticals form a geometrical progression.

33. A strip of carpet one-half inch thick and $29\frac{1}{2}$ feet long is rolled on a roller 4 inches in diameter. Find how many turns there will be, remembering that each turn increases the diameter by one inch, and taking as the length of the circumference $\frac{22}{7}$ of the diameter. (HARVARD)

34. Solve $\sqrt{x} - \sqrt{y} = 2$, $(\sqrt{x} - \sqrt{y})\sqrt{xy} = 30$. (YALE)

35. Solve $x - y - \sqrt{x - y} = 2$, $x^3 - y^3 = 2044$. (YALE)

36. Find the middle term of the expansion of $(x - y)^{20}$. Express this term in its simplest form when $x = \left(\frac{2^{-\frac{1}{2}}}{3^{-\frac{1}{2}}}\right)^{\frac{1}{10}}$, and $y = \sqrt[4]{\frac{3^{\frac{1}{2}}}{2^{-\frac{1}{2}}}}$. (HARVARD)

37. Extract the square root of $38 - 12\sqrt{10}$. (REGENTS)

38. Solve $\sqrt{5x + 1} = 3 + \sqrt{x + 2}$.

39. Find the sum to infinity of the progression

$$-8, -\frac{2}{5}, -\frac{1}{50}, \cdots$$ (MASS. INSTITUTE)

40. Solve for x $\qquad 10^{2x} - 10^x - 2 = 0$.

41. Find the value of the repeating decimal $.3727272 \cdots$ in fractional form.

42. Solve $3(x + 1)^{\frac{2}{3}} + x^2 + 3x^2 + 3x + 3 = 0$.

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